Geometry in the 19th Century
A study in five papers

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Le secret d'ennuyer est celui de tout dire.[1]

Introduction

Alfred North Whitehead said that "the safest general characterization of the European philosophical tradition is that it consists of a series of footnotes to Plato".[2] Similarly, one might argue that geometry, at least until the start of the 19th Century, consists of a series of footnotes to Euclid. That might be a little harsh on Descartes, who introduced the use of algebra into geometry, or on Desargues, who could be considered the founder of projective geometry, but nonetheless this is a view which has much to recommend it.

Consider next Figure 1, a product of the end of the 19th Century. The Italian, Gino Fano, introduced his famous plane ([5]) in 1892, and it contains exactly seven lines and seven points. As Gray notes ([7] p.264), this is introduced as a counter-example, rather than an object interesting in itself. However, at the start of the century this would hardly have been considered a geometric object at all - how could a "line" contain only three points? Even the perceived reason for producing it (showing that an axiom is needed to prevent the fourth harmonic point of A, B, C being just C) would have seemed vacuous. Somehow the whole conception of what constitutes "geometry" has changed unrecognisably.

In this essay we will discuss some of the key changes in the *concepts* of geometry which have led to this massive reevaluation. We focus on concepts, rather than methods. We will do this by discussing just five influential and ground breaking papers; Poncelet's *Traité des Propriétés Projectives des Figures* ([16]) of 1822; Lobaschevskii's *Geometrische Untersuchungen zur Theorie der Parallellinien* ([13]) of 1840; Riemann's *Habilitationsvortrag* ([17]) of 1854; Klein's Erlangen program address from 1872; and finally Hilbert's

[1]Voltaire "Sixiéme discours: sur la nature de l'homme," Sept Discours en Vers sur l'Homme (1738).

[2]([18] p.39)

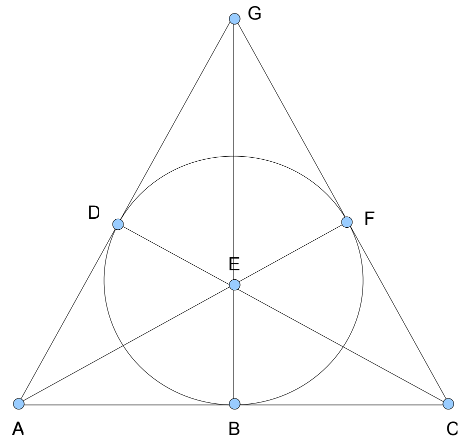

Figure 1: Fano's Plane

Grundlagen der Geometrie ([9]) of 1899. We are not claiming there are not other works equally deserving mention - one has to feel a little uncomfortable omitting Poincaré, for example. However these five works, spread across the century, are both central to, and characteristic of, the revolutionary changes during the period, and illustrate the major shifts in the perception of what defines geometry.

Shaking the Foundations: Poncelet's Traité

"*This work is the result of researches undertaken during the Spring of 1813, while in a Russian prison. Deprived of all books and assistance, and distracted by the misfortunes both of my fatherland and of myself, I was unable to bring them to the right level of perfection. However, I have since found these theorems fundamental to my work, which is to say the principles of central projection of figures in general and conic sections in particular, properties of secants and tangents common to these curves, properties of their inscribed and circumscribed polygons, and so on*".[3]([16] preface to the first

[3] Author's translation.

edition)[4]

So begins one of the most remarkable but least well known intellectual achievements of its time. Remarkable because, despite[5] his difficulties, Poncelet built a novel conception of geometry.

His starting point is the problem of the differences in power between analytic geometry, which uses coordinates to abstract figures into numbers, and "synthetic" geometry, such as Euclid's, which works from "curves, lines, angles and areas, and algebra is avoided" ([7] p.18). The former is very potent - once one has numbers one can manipulate these without regard to physical significance. The latter, however, is highly constrained - a line with three points A, B, C may need to be treated differently depending on which lies between the others.

Poncelet's resolution combines three transformations of a figure so that if a theorem is true beforehand, it remains true afterwards. The first is *central projection*. Figure 2 is one of his figures, and we see points A, B, C, D centrally projected from S to points A', B', C', D'.

Immediately a key feature becomes apparent; since Poncelet will treat these sets of points as fundamentally equivalent he cannot be concerned about *length* or *angle* because these are not preserved. However, he demonstrates ([16] p.12) that some things are constant. In this example, he has arranged A, B, C, D in *harmonic ratio*, i.e. $\frac{CA}{CB} = \frac{DA}{DB}$, and he shows that this implies that $\frac{C'A'}{C'B'} = \frac{D'A'}{D'B'}$. Harmonic ratio, then, is a property which is meaningful in his geometry.

This is not entirely new - Monge projected lines in a somewhat similar way in his descriptive geometry - but Poncelet uses projection to new ends. For example, he shows that projections take one conic to another. So, provided

[4] All the works studied here are available on the Internet. Sadly, Poncelet's work is only available in French. Additionally, since it runs to nearly 500 pages and 37Mb (he is not a man of few words and his introduction alone runs to 32 pages), a close reading has proved impossible.

[5] Or, maybe, because. Perhaps his isolation allowed his creative mind to run in directions which interaction with others might have constrained.

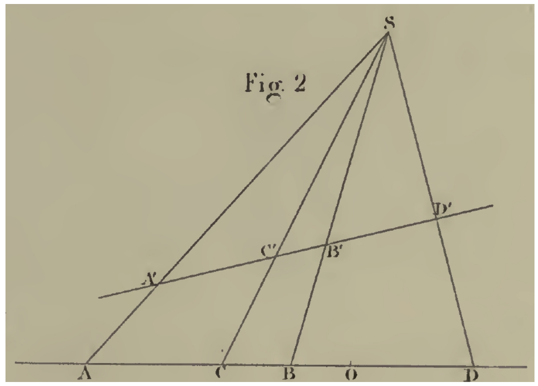

Figure 2: Poncelet's central projection

he considers only properties preserved by projection, he can prove theorems with circles, and then generalise these to theorems about any conic simply by arranging a projection.

The second is the notion of *pole and polar*. We use another of his figures, Figure 3, to see this situation, discussed earlier.

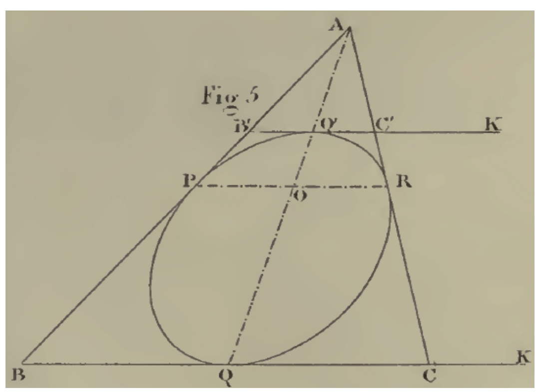

Figure 3: Poncelet's concept of pole and polar

Starting with a fixed conic (here an ellipse), a point (A - the pole) can be transformed into a line (PR - the polar) by joining the points where the tangents from A meet the conic. The reverse also applies and PR can be transformed to A; a simple extension covers the situation where the line fails to meet the conic.[6] Poncelet shows that this transformation has some interesting properties; for example, if three points are collinear, then their polar lines are coincident. So, theorems about coincidence and collinearity are equivalent.

The third transformation was the more controversial one of *continuity*. Two systems are equivalent if one can be obtained from the other by "imposing on certain parts of the figure a continuous but otherwise arbitrary movement".[7] The lack of precision in the definition should have been a warning that this was going to cause trouble. Equally problematic was his new and highly confusing definition of what it means for two curves to meet; something he has to propose, because he wants to be able to say that two conics, tangent at two points are equivalent to two concentric circles (which clearly don't "meet" at all in the normal sense). His solution did not go down well.

Poncelet's influence on geometry was immense,[8] and one could argue there were two reasons for that. The first is that the sheer originality of his work forced geometers to think in new ways. By dispensing with metric concepts, for example, Poncelet brought a new approach to thinking about what features of a figure were important. Moreover, by showing an equivalence between points and lines he paved the way for a much more abstract view of the subject; one could no longer intuitively think of a line in the same way if, somehow, it was interchangeable with a point - or in three dimensions with a plane! Secondly, and ironically, the lack of rigour and hence gaps in his reasoning opened up numerous lines for future research; for example, how could one dispense with the fixed conic which seem to be required to create polarity; how would polarity deal with curves of degree higher than two; how

[6] All this also is not entirely new - as Gray notes ([7] p.21) it was discovered by Apollonius taken up by Brianchon.

[7] Quoted in ([7] p.19).

[8] Despite the initial reception from analysts, such as the highly influential Cauchy, who baulked at the lack of rigour.

might sense be made of his idea of "meeting"?[9] Poncelet had opened the door to a whole new world of geometry.

Working in Parallel: Lobaschevskii's non-Euclidean Geometry

Lobaschevskii's work might easily seem less consequential than the others. Where Riemann, say, challenges the foundation of the subject, Lobaschevskii challenges solely Euclid's fifth postulate.[10] Unlike Euclid's other propositions, this was rightly seen as problematic by generations of mathematicians, who tried, vainly, to establish it from the others, or prove its intuitive nature. The thrall Euclid held on geometry is well documented, and we content ourselves with one small example. Hilbert in ([9] p.12) writes "it is possible to deduce the following simple theorem, which Euclid held - *although it seems to me wrongly* - to be an axiom."[11] Even the man who "many would say was the leading mathematician in the world of his generation" ([7] p.251) couldn't quite bring himself to say outright that Euclid was wrong. Lobaschevskii may be addressing a single point, but he boldly lays an axe to the root of Authority.

Written from his intellectually isolated position in Kazan, Russia, "Geometrical researches on the theory of parallels" ([13]) was Lobaschevskii's third major attempt to communicate his ideas on what was later called "non-Euclidean" geometry. He had had no success previously in Russian or French, and would fail now in German. In his lifetime only Gauss realised its significance.

We highlight three crucial aspects of Lobaschevkii's approach. The first is his new definition of "parallel" (§16), see Figure 4. From a point A on

[9] Sadly, this last challenge was not taken up!

[10] "If a right Line, meeting two others, make two interior angles on one side of it together less than two right angles : these two Lines, produced if necessary, will meet on that side." ([4] p.9).

[11] Author's italics.

one side of a point D on a line BC, some lines cut DC (e.g. AF), others miss (e.g. AK'). He defines as parallel the boundary case between the two (AE); a definition which makes sense whether the geometry is Euclidean or not. This is the same as that the Bolyais used, also working in isolation in Hungary; there is significant overlap between their work and Lobaschevskii's. Having shown his definition meaningful, he demonstrates that, whatever the geometry (subject to his initial assumptions), the sum of the angles of a triangle is either always π or always less than π. The former is well known, the latter is his new "Imaginary" Geometry.

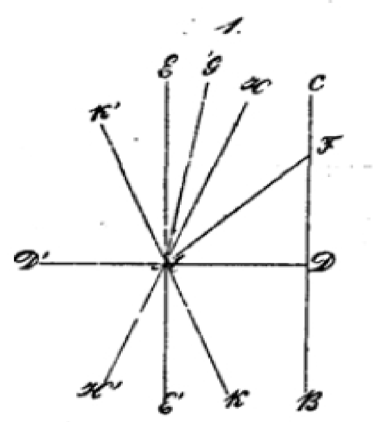

Figure 4: Lobaschevskii's Definition of Parallel

Lobaschevkii's second big step is to take his definition seriously, and boldly follow the implications through. This is what really sets him apart from predecessors such as Saccheri and Legendre, each of whom reached conclusions they felt were contradictory and stopped.[12] Lobaschevskii's boldness is apparent in his figure (Figure 5); somehow CD, however extended, fails to

[12]Though, to be fair on Saccheri, this was what he set out to do. His problem was making the wrong initial assumption - that the parallel postulate was valid.

meet AB. A contemporary reader might have stopped here; surely the author is mad.

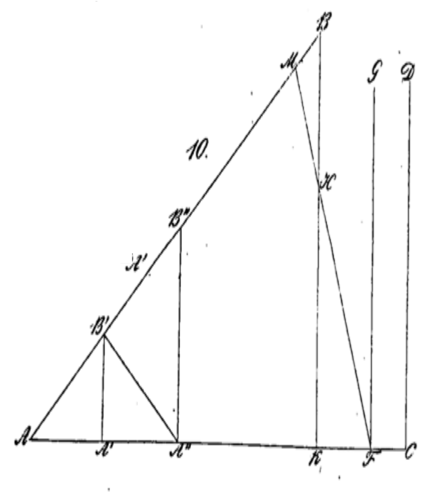

Figure 5: Lobaschevskii's Figure 10

None of this is entirely new, but now Lobaschevskii takes one final and vital step, by moving into the third dimension. First he produces his oricycle (§31), a two dimensional curve with properties similar to a Euclidean circle. Then he rotates his oricycle around a point to produce a surface in three dimensional space, his orisphere (§34). Now he demonstrates real virtuosity: by switching between plane Euclidean and Spherical geometry, he proves that geometry on the surface of the orisphere is Euclidean. This means he can take trigonometric formulae off the orisphere, and map them down to trigonometric formulae in his non-Euclidean plane. In this way he reveals three intimately related geometries - the well-known Euclidean and Spherical geometries, and his "Imaginary" geometry.

Lobaschevskii closes with these trigonometric formulae and we note two reasons why these are important. Firstly, he is intent on describing space, which previously had always been tacitly assumed Euclidean. These formulae give a mechanism for checking how geometry works in space - measure the angles and sides of a triangle and plug in the numbers. A sufficiently large triangle might be obtained through "astronomical observations", and with good enough data (not available at the time) one could decide if space was Spherical, Euclidean or "Imaginary". Secondly, as he demonstrates, his formulae simplify to the familiar Euclidean formulae as the size of a triangle tends to zero - space on the small scale will appear Euclidean. Not only that, but if we put $a\sqrt{-1}, b\sqrt{-1}, c\sqrt{-1}$ for the sides, we obtain the then familiar formulae of spherical geometry. The fact that these values are complex explains his choice of term "Imaginary".

Lobaschevskii's words fell on deaf ears. However after his death, the disc models of Beltrami and Poincaré led to a reevaluation. Gray has noted,([15] p.54) two key factors about the discs. Firstly, they lack singularities and so counter objections to "the anomalous nature of known surfaces of constant negative curvature" - which had been a major criticism of trying to do geometry on objects such as Minding's surface ([7] p.123-124). Secondly, they provide a robust model in which Euclid's parallel postulate is false, but all other Euclidean assumptions hold. There is nothing self-contradictory about denying the parallel postulate, and Euclidean and non-Euclidean geometry stand or fall together. Geometers now had to face the fact that there was no single "true" geometry, and no *a priori* geometry of space.

Dismantling the Scaffolding: Riemann's Habilitationsvortrag

In 1854,[13] as part of his Habilitation, Bernhard Riemann gave his Habilitationsvortrag, a lecture within the Philosophy faculty. He had prepared three topics, and "Herr Geheimer Hofrath" Gauss chose "On the Hypotheses Ly-

[13] Biographical details from Gray ([7] p.189-190).

ing at the Root of Geometry" ([17]).[14] The paper is hardly lucid, though one should recognise this was a lecture not a paper, and there was at least one influential non-mathematician in the audience. Riemann's professional career hangs on this, and he needs to be even more careful than Gauss to avoid the "Geschrei der Böotier". Any reference to non-Euclidean geometry needs to be very cautious.

Riemann's goal is to challenge the most fundamental assumption of geometry. Previously the objects of geometry - lines, curves, points, etc. - had been quietly assumed to sit in some neutral ambient space (the "scaffolding") while geometers got around to studying their properties. Moreover, there was assumed to be a very strong correspondence (if not identity) between ambient space and real space. Radically, Riemann questions this. He elects to start with ambient space, and then demonstrate how this causes geometric objects to acquire their properties.

Riemann starts by defining space as an "n dimensional manifoldness". He shows how to build manifoldnesses of increasing dimension, by allowing a point to sweep out a line, a line to sweep out a surface, and so on. Next he shows how to break down a manifoldness into smaller and smaller dimensions *when this is possible*[15] thereby obtaining a set of coordinates for points within the manifoldness. Some reasonable assumptions are made - the manifoldness should be smooth, should have a "measure-determination" independent of position, and should be "flat in the smallest parts" (i.e. be locally Euclidean) - and this enables Riemann to constrain the formulation of the metric - the distance between points. Note that the manifoldness does not inherit any properties from the space (Euclidean or otherwise) it sits in; its properties are necessarily implicit (like Gaussian curvature) because there is no other space. Straight lines are defined as geodesics on the surface of the manifoldness.

[14] Author's translation. Clifford's title seems - like much of his translation - a little unwieldy.

[15] These italics appear in Clifford's translation, but not in the original German. One wonders how italics were delivered in a lecture!

Riemann discusses some specific examples of the many possible geometries derived from different manifoldnesses. Manifoldnesses with constant curvature are strongly appealing because only in these can a fixed shape be moved around; one suspects that Riemann couldn't quite believe physical reality might not come into this category. Constant curvature of zero is Euclidean, but this is just the special case of any space of constant curvature α in which the metric is

$$ds = \frac{1}{1 + \frac{1}{4}\alpha \sum x^2} \sqrt{\sum dx^2} \ .$$

As Gray notes ([7] p.192), there is a covert nod here in the direction of non-Euclidean geometry, which appears when α is negative.

The question of whether physical space is Euclidean becomes completely subverted. Perhaps it is, perhaps not: we can determine this by taking measurements and working out if the curvature is always zero. But in neither case can we assume anything about the nature of geometry: there are an infinite number of geometries, each built on a different manifoldness, and although one might correspond to the manifoldness of physical space, it does not have any conceptual uniqueness.

Riemann also brings a new angle on the synthetic versus analytic argument. Since the manifoldness comes first, and coordinates and measure-relations on the manifoldness can only be defined algebraically, then algebra *inevitably* comes first. Once a geometry is defined within the manifoldness, we can attempt to work synthetically, but the priority must be algebraic.

Riemann's paper was almost entirely unprecedented; apart from a little help from Gauss and Herbart, as he notes himself, he had little to build on. Initially his work went almost unnoticed - the difficulty of the concepts, the lack of mathematical formulae, and the obscure language did not assist an easy reception. In the longer term, though, his work was deeply influential, and we highlight three aspects.[16] First, as Gray notes ([7] p.219) we see the influence on Beltrami (and on Poincaré), who used the metric concept to build his disc model, and so satisfy those who questioned Lobaschevskii's work. Secondly, there is a profound influence on physics: by 1870 Riemann's English translator William Clifford in his address to the Cambridge Philosophical Society is discussing curvature and distortion of space.[17] The culmination of these ideas comes in Einstein's Theory of General Relativity. Finally, and most importantly, Riemann increased the number of geometries from a handful, of which perhaps one held a privileged position, to an infinite number, each defined by a manifoldness which need not even be continuous. He even, vaguely, gives a hint how this zoo of geometries might be categorised; with "the postulate that certain given things are to be regarded as equivalent". Surely this is at least a rough indication of the direction Klein will next map out.

A New Superstructure: Klein's Erlangen program

A Modern Example

In this section we consider an example of how a *group action* can generate a geometry, and help us understand and analyse its symmetries. We are going to work towards a non-Euclidean geometry which Klein created in 1879. However, we should note in advance that this is a different and more modern approach than the one taken by Klein ([11]), and Klein himself did not comment on the non-Euclidean nature of his geometry; as Gray notes ([6] p.129), Klein preferred projective geometry and was interested in "using group theory to get at the [projectively] invariant configurations".

[16] There are many more which Riemann just hints at.

[17] Quoted in http://clifford-algebras.org/monty/MONTY.pdf p.4

Let's start with a simple example.[18] Consider the group $G = \mathbb{C}$ under addition, and we will have this act on the complex plane $\mathbb{C}$ by the action:

$$w(z) \longmapsto w + z$$

It is fairly obvious that this is indeed a group action,[19] and it is faithful.[20] We will note also that this group action clearly preserves a Euclidean distance defined by $d(z_1, z_2) = |z_1 - z_2|$.

This isn't, however, a very interesting group action. For example, the orbit of a complex point, defined by $O(z) = \{g(z) : g \in G\}$ is just the whole complex plane. We need to tune our group G a little. Consider then the homomorphism which takes just the fractional part of the real and imaginary parts of a complex number[21]

$$f : G \to G' = \{x + iy \in \mathbb{C} : 0 \leq x, y < 1\}$$

where

$$f(x + iy) = \text{frac}(x) + i\ \text{frac}(y)$$

Because this is a homomorphism,[22] the kernel $\Gamma = \ker f$ is a normal subgroup of G.

Let's consider what this kernel is; it's the preimage of 0, and so it is the Gaussian integers, i.e. those complex numbers whose real and imaginary parts are integers. Accordingly, the orbit of a point z under this group will be the set of all complex numbers which differ from it by a Gaussian integer.

We now create the orbit space $\mathbb{C}/\Gamma$ as the set of all these orbits; naively

[18]Our approach, perhaps slightly non-standard, has been influenced by Gray's paper ([6]), Klein's paper ([11]), Berger ([1] p.4-11) and Baez's excellent and intuitive account at http://math.ucr.edu/home/baez/week214.html.

[19]Because as a mapping $w_1 + w_2$ has the same effect as the mapping w_1 followed by w_2.

[20]The only element of G which maps each z to itself is the identity 0.

[21]We're using the "fractional part" function, frac(x) $= x - \lfloor x \rfloor$ e.g. frac(1.24) = 0.24, frac(-1.24) = 0.76. The definition has to be precise to ensure this really is a homomorphism.

[22]Note that the binary operation on G' is addition mod 1.

we have “glued” together all complex points which differ only by a Gaussian integer. We can get a picture of this space by noticing the obvious 1-1 mapping between the orbit space and the “square” $\{x + iy : 0 \leq x, y \leq 1\}$, provided we identify the opposite sides of the square as indicated in Figure 6 (for example, the points 0.4 and (0.4 + i) are equivalent).

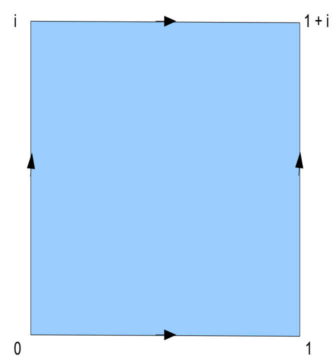

Figure 6: The Flat Torus

Note the intentional labeling of this as a “flat” torus. Although we can imagine bending up the sides and “glueing” them in $\mathbb{R}^3$ to create a standard torus, if we just leave the sides unbent but identified as indicated, the geometric nature of the torus is obvious but the metric, respected by the group action, is Euclidean. We can also verify that the genus is indeed 1 by using a simple triangulation (see Figure 7), and Euler's formula $V - E + F = 2 - 2g$. Here we have $V = 9$, $E = 27$, $F = 18$, hence since $V - E + F = 0$ the genus g must be 1.

Finally, we can see how our group G' acts as the group of symmetries of the flat torus. The symmetry group is continuous, and acts by “rotating” our flat torus in a combination of two independent directions; parallel to the Real and Imaginary axes.

Now, we will produce Klein's more complex geometry. We will use exactly the same process as above, apart from:

1. We will use the group $PSL(2, \mathbb{Z})$ as our initial group

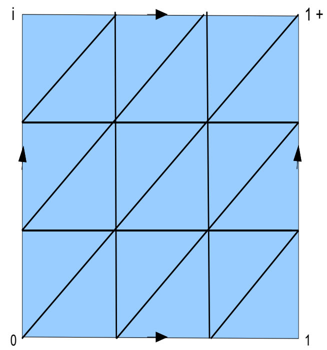

Figure 7: A triangulation of the flat torus

2. This group will act on the upper half plan $H = \{z \in \mathbb{C} : Im(z) > 0\}$
3. We will use the natural homomorphism $h : PSL(2, \mathbb{Z}) \to PSL(2, \mathbb{Z}/7\mathbb{Z}) = G_{168}$ to produce an interesting kernel
4. Finally, as before, we will consider the orbit space $H/\ker h$ and the symmetries of this created by G_{168}

Firstly, then, consider the group $PSL(2, \mathbb{Z})$ which consists of 2×2 matrices under multiplication, with integer entries, with determinant 1, and with two matrices considered equivalent if one is -1 times the other.[23] We will have this act on H by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) \longmapsto \frac{az + b}{cz + d}$$

This is indeed a group action,[24] and it is faithful. It also has an interesting feature discovered by Dedekind in 1878 ([6] p.116). He showed that every element of this group copies the set $R = \{z \in \mathbb{C} : |z| \geq 1, -1/2 \leq Re(z) \leq$

[23]This latter is the only difference between $PSL(2, \mathbb{Z})$ and $SL(2, \mathbb{Z})$, and is required purely to ensure the action is faithful

[24]A little algebra is sufficient to justify this actually quite remarkable result

$(z)\}$ to a different copy, with no overlap apart from at the boundaries where a little care is needed. We can imagine H as made up of an infinite number of copies of R each labeled by an element of $PSL(2,\mathbb{Z})$. Figure 8[25] shows R and its copies tiling the plane. Notice how each copy has a vertex either at infinity, or on the real line.

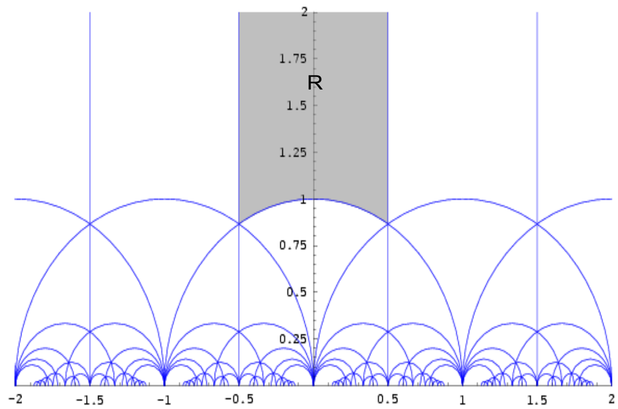

Figure 8: The upper half plane tiled by copies of R

We can also note here that these mappings respect the non-Euclidean metric in H

$$d(z_1, z_2) = \log \frac{|z_1 - \overline{z_2}| + |z_1 - z_2|}{|z_1 - \overline{z_2}| - |z_1 - z_2|}$$

and the geodesics are circles which meet the real line at right angles (which includes lines parallel to the imaginary axis). In other words, the boundary of R and its copies are all geodesics. Plus, with this metric all the copies have the same area.[26] Sadly we don't have space to say anything about the interesting geometry induced already.

Now, as before, we want to cut our group down in size. The homomorphism h above takes all the entries in a matrix and reduces them mod 7. As

[25]From http://en.wikipedia.org/wiki/Modular_group (towards the end).

[26]From http://en.wikipedia.org/wiki/Poincaré_metric though these statements are easy enough to check algebraically.

before, we want to consider the kernel of this homomorphism, Γ_7, and its action on H. Γ_7 contains matrices congruent to the identity mod 7, and its action is best understood by considering G_{168}. This group has 168 elements, as shown by Gray ([6] p.117),[27] and so Γ_7, instead of filling the whole H with copies of R, fills exactly $\frac{1}{168}^{\text{th}}$ of the plane. In other words, to fully tile H we need to start with 168 carefully chosen copies of R. The orbit space H/Γ_7 can then be seen to be H with all the orbits of each copy "glued together".

Visualisation of this becomes difficult because of the fact that R has one vertex at infinity, so we will simplify our picture by mapping H into the unit disk $|z| < 1$ in such a way that the real axis goes to $|z| = 1$. As Gray notes ([6] p.118), Klein seems have used the Riemann mapping theorem to assume the existence of such a mapping, and it was left to his student M. W. Haskell to find one explicitly. This gives rise to Figure 9.[28]

Each copy of R in this image corresponds to a pair of triangles - one coloured and one grey - and we have highlighted one of these. The copies of R are grouped into 24 heptagons, each containing 7 copies of R, so 168 in total as required. Note that some of the heptagons are broken up on the boundary - it is the action of Γ_7 which identifies the edges by gluing together sides 1 and 6, 3 and 8, 5 and 10, 7 and 12, 9 and 14, 11 and 2 and 13 and 4. What we see here, in fact, is a tiling of the Poincaré disc, and these triangles really are triangles because the geodesic sides of R have been mapped into geodesics in this picture. The geometry is certainly not Euclidean!

Now, let's calculate the genus of this object. As Gray notes ([6] p.119), we have a triangulation in front of us, with 336 triangles, 336.3/2 edges (because the three edges of each triangle are counted twice) and each triangle has one vertex at which 14 meet, one at which 6 meet, and one at which 4 meet, making $336/14 + 336/6 + 336/4 = 164$ vertices. Hence $V - E + F = 336 - 504 + 164 = -4 = 2 - 2g$, and so g the genus, is indeed 3.

[27] Klein just states this!

[28] A nice image from http://www.valdostamuseum.org/hamsmith/JBGEKQ.html (towards the end).

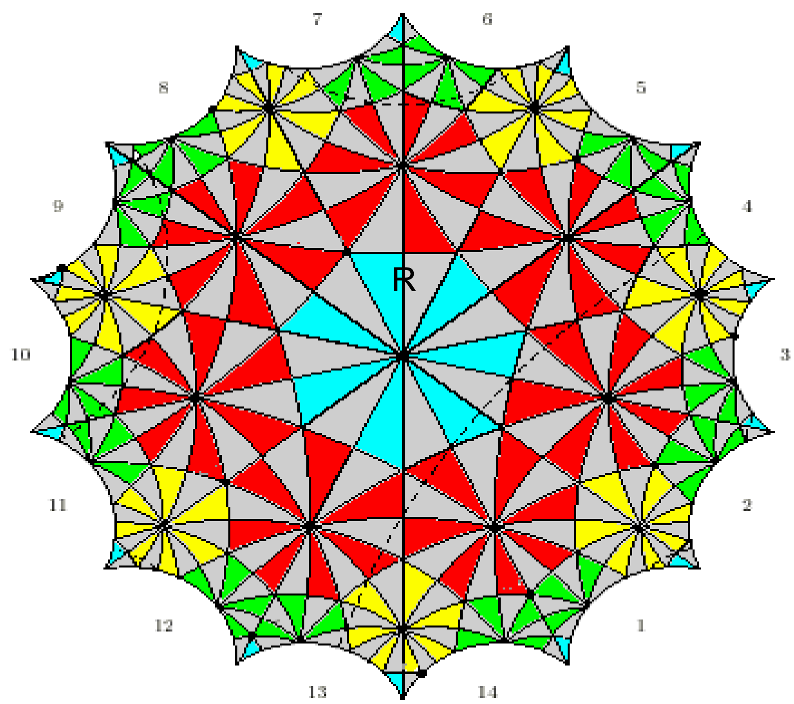

Figure 9: The induced tiling of the disk

Finally we get on to the issue Klein was exploring - the symmetries of this object, described by the group G_{168}. Unlike the flat torus we saw earlier, this group is not continuous; there are exactly 168 symmetries, and Klein in his paper ([11] p.290) takes time to explore them, using the tools of elementary group theory. For example, there are 14 subgroups "each of four elements such that every element different from the identity has period 2" ([11] p.290), i.e. the famous Klein 4-group. Remarkably, G_{168} is also the symmetry group of the Fano plane we saw earlier - but that is a different story.

The Historical Background

In this section we will look to situate this group action in a historical context. Accordingly, we will review firstly Klein's Erlangen program,[29] which set up a framework for using the theory of groups to understand geometry, and then as a specific example briefly review Klein's paper of 1879 in which he introduced the example considered above.

Felix Klein had a singular early mathematical life.[30] At 16 he became Plücker's assistant, and completed the latter's posthumous geometrical work. He worked closely with the algebraic geometer Alfred Clebsch, and then moved to Berlin where he first met his friend Stolz, and was not highly regarded by the influential Weierstrass who called him "anathema maranatha".[31] He met the Norwegian Sophus Lie, and traveled with him to Paris where they studied the work of Galois. By the time he went to Erlangen, at Clebsch's recommendation a full professor at the age of 23, he had a impressively holistic understanding of the geometry of his time - both the French and German schools, non-Euclidean geometry he had discussed with Stolz, and the less well known works of Riemann and von Staudt he had studied himself. As a second string in his bow, he had the work done with Lie on the group concepts of Galois and Jordan.

As a new professor, Klein had to give an inaugural address, and "Vergleichende Betrachtungen über neuere geometrische Forschungen" ([10]) was the result. Like Riemann's Habilitationsvortrag it is written as an address, though in practice it was not given as such but circulated as a text, and so lacks mathematical formulae. It is clearly the work of a bold, innovative thinker, not too concerned with painstaking detail, determined to break new ground, almost

[29]Which is a challenging proposition since Klein himself notes that it can "hardly be presented in a more concise form" ([8] introduction).

[30]Biographic details from ([7] p.222-224) and ([19] p.212-215).

[31]Klein excited strong options. Young notes that "his teacher Carathéodory" claimed Klein "had no idea how to *prove* things, he could just *see* that they were true". There is an echo here of Gordan's comment that Hilbert's work was "not mathematics but theology" ([7] p.252).

wholly intuitive.

Klein could see how geometry had become fragmented; he was familiar with, for example, projective geometry, Euclidean and non-Euclidean geometry, inversive geometry, and Lie's sphere geometry. He was also aware that geometers had become used to defining configurations as equivalent - for example under a dualistic transformation - without there being a clear framework for understanding what this really meant. From his work with Lie, he had found a way to use the concepts of group theory to tie this all together into a more abstract, but unified, setting.

Klein begins ([8] §2) by considering space - or any manifoldness - and a (loosely defined) group of transformations of space to itself. Geometrical properties of a configuration should not change when it is moved around, and so - *vice versa* - geometric properties are precisely the properties of space which are not changed by (are invariant under) the group. Thus Klein's restatement of the object of geometry is "Given a manifoldness and a group of transformations of the same; to develop the theory of invariants related to that group". Geometric objects are then exactly those invariant objects; for example, it does not make sense to talk about circles in projective geometry - instead we should deal only with conic sections, because these are invariant under projective transformations.

With this deceptively simple concept in place, Klein offers three ways it can be used to investigate the relationship between geometries. Firstly, we can consider moving from a manifoldness and group of transformation, to the same situation but with some configuration fixed. This is equivalent to choosing a subgroup of the original group - specifically those transformations which leave the configuration invariant. This is the situation we met in the previous section - we started with H and $PSL(2,\mathbb{Z})$ and saw that fixing the configuration $H/\ker h$ was equivalent to using the subgroup G_{168}. Mathematically this is important because not only can new geometries be discovered "hidden" within broader structures, but also the detailed apparatus of group theory can be used to analyse them.

Secondly, we can move in the reverse direction from a group to a larger group, in which case the invariant configurations become coarser. Klein ([8] §3) gives the example of the move from Euclidean to projective geometry; the former is just space (2 or 3 dimensional) acted on by the group of "motions". The latter becomes apparent when we enlarge the group to include the projective transformations. We can then extend further, by including "dualistic" transformations, and then "imaginary" ones. The notion that Euclidean geometry is contained within Projective geometry now becomes clear in an exact manner.

Finally, we can see when apparently different geometries are fundamentally identical. Suppose we have a manifoldness A and a group B, which can be converted - in some way, Klein is hardly rigorous here! - to another manifoldness A' and group B'. The "the method of treating A with reference to B at once furnishes the method of treating A' with reference to B'" ([8] §4). He gives an elementary example; let A be a straight line and B the linear transformations of A to itself. Let A' be a conic section, and establish a (1-1) correspondence from A to A' by projecting from any fixed point on A'. Then B' is the group of linear transformations which fix the conic, and these two geometries are equivalent. Later (e.g. ([8] §7) where he discusses the Sphere geometry of Lie, which Darboux also discussed) he considers much more complex examples, but the importance of his technique is clear; we suddenly have a mechanism for proving the equivalence of geometries, and hence carrying the results in one across into the other.

The only thing which seems to be missing is how to incorporate Riemann's notions of a metric on the manifoldness. Klein can create metrics - by considering how a system corresponds to a fixed configuration - but it is not clear that he really can cover the full generality of Riemann's approach. The resolution to this had to wait for Élie Cartan.[32]

[32] According to http://en.wikipedia.org/wiki/Erlangen_program (first paragraph).

Klein closes with a set of Notes,[33] some more valuable than others. Gray ([7] p.228) highlights Note V, wherein Klein discusses how the projective metrical geometry he discusses as "projective geometry of the plane with reference to a conic" is equivalent to "the so-called non-Euclidean Geometry". Lobaschevskii's wild beast has been firmly tamed. This author cannot help but highlight the visionary nature of Note IV. Given modern conceptions of physical space, to have considered as early as 1872 that nobody can "be prevented from [...] claiming that space really has four, or any unlimited number of dimensions, and that we are only able to perceive three" seems strikingly bold.

The immediate aftermath of his paper was muted, but, as Gray notes ([7] p224,228), "the Erlangen programme became in the 1890s a retrospective guideline for his research .... uniting geometry and group theory, two subjects that had not only progressed considerably in the intervening 20 years but had indeed grown closer together." It is an open question, though, whether Klein's paper was directly of great influence, or whether the work of Klein, Jordan, Lie and even Poincaré brought these two subjects together quite irrespective of these early remarks. In a mathematical sense this is a moot point.

Klein's work of 1879 ([11]) is a difficult but virtuoso performance, drawing together geometry, algebra and complex analysis. Consideration of the seventh order transformation of elliptic functions leads him to "those linear substitutions $\frac{\alpha\omega+\beta}{\gamma\omega+\delta}$ that are congruent to the identity modulo 7"; in other words, what we earlier called G_{168}. Having discussed this group in considerable - but elementary - detail, he is able to find numerous geometries generated by the action of the group, and their exquisite symmetries and invariants which he directly relates to properties of the original G_{168}. We have already noted the geometry which emerges from this action on the upper half complex plane. Klein also discusses the representation of this as a complex

[33] Those these don't seem to have been included in the initial publication

curve - his demonstration that this can be seen in complex homogeneous coordinates as $\lambda^3\mu + \mu^3\nu + \nu^3\lambda = 0$ shows a slick proficiency in the methods of Plücker and Möbius. The invariants of the group then gives rise to "24 inflection points, 56 contact points of bitangents, and 84 sextatic points" on this curve. The fecundity of his method is exceptional, although often hard to follow. We also get to see[34] the symmetries of the 28 (=56/2) bitangents; the geometry as the surface of three intersecting hyperboloids; later authors added the action of G_{168} in a different guise as $SL(3;\mathbb{Z}/2\mathbb{Z})$; and the 27 lines on a cubic surface. In a sense it is ironic that Klein's efforts to unify geometry can actually lead to an explosion in equivalent geometric objects.

A Conclusion: Hilbert's Grundlagen

We have seen a thread through the 1800s wherein past assumptions are shaken, and new structures emerge. Mathematicians were left with the uncomfortable realisation that nothing, even in a fundamental subject like geometry, was certain. As Gray delightfully puts it ([7] p.247) "signs of anxiety about the nature of geometry run like fissures through late 19th-century mathematics".

Hilbert was not the first to seek to address this. Previous attempts came, for example, in Pasch's *Vorlesungen uëber neuere Geometrie* of 1882, and Veronese's *Grundzüge der Geometrie* of 1894. Hilbert, being Hilbert, is more authoritative than his predecessors. Moreover, the timing of his work, right at the end of the century, forms a fitting conclusion to our overview.

Hilbert's work ([9]) is extremely readable, without the pedantry of, say, *Principia Mathematica*, or the lack of intuitive clarity of Tarski's more rigorous axiomatisation in the 1920s.[35] He wishes to show that it is *possible* to axiomatise geometry, but doesn't feel a need to resolve every minor point.

[34] Gray([6]) is particularly helpful in teasing these out.
[35] See http://en.wikipedia.org/wiki/Tarski's_axioms throughout.

Instead we see a broad discussion of how an axiomatic approach works, the relationships between axioms, and the various geometries which apply when different axioms are accepted.

He starts with three intentionally empty concepts; points, lines and planes.[36] As Gray notes,([7] p.255) Hilbert has not forgotten his famous remark about beer-mugs.[37] Their properties are defined through groups of axioms, which are offered as a smörgasbord from which, by careful selection, new geometries can be created:

1. Axioms of *Connection*; what does it mean for two points to be collinear, for two lines to meet?

2. Axioms of *Order*; what does it mean for a point to be between two others?

3. The axiom of *Parallels*; Hilbert's work comes from a set of lectures on Euclidean geometry, so he will work primarily incorporating this axiom. It is clear, however, that his *approach* is applicable to any geometry.

4. Axioms of *Congruence*; what does it mean for two segments to have the same length? This is important to Hilbert because much of his work will depend on combining segments by addition and multiplication.

5. The axiom of *Continuity*, also known as Archimedes axiom; this is the most complex, and the one which when not included leads to the most technical and counter-intuitive geometries.

6. The axioms of *Completeness*; this axiom is unused, and was not present in the original edition.[38] It states that there is no way a new element or concept can be added to a system of points, lines and planes to form

[36] Note how this reverses Riemann's approach.

[37] "One must be able to say at all times – instead of points, straight lines, and planes – tables, beer mugs, and chairs."

[38] This axiom came under heavy criticism from Frege, see ([7] p.328).

a new geometry which still meets the above axioms. Although there is no further discussion, that Hilbert was sensitive to the fact that this was needed shows how far his thinking has progressed.

Importantly, Hilbert recognises he must show these axioms are *consistent* and *independent*. His approach to the former is remarkable. He produces an arithmetic domain Ω and shows how this meets all axiom groups 1-5 above, which implies that "from these considerations, it follows that every contradiction resulting from our system of axioms must also appear in the arithmetic related to the domain Ω." Euclidean geometry and arithmetic stand or fall together.[39]

His approach to independence is to build consistent geometries which meet only certain axioms. For example, he demonstrates the significance of the axiom of Archimedes by producing a "non-archimedean" geometry meeting all axioms apart from 5. The example is indeed "convoluted",[40] but that is not the point. Hilbert shows the axiom is needed for Euclidean geometry, and interesting mathematics can be done by negating it. He misses some fine points - for example, E. H. Moore showed in 1902 ([14]) that Hilbert's use of Pasch's axiom (II, 4) was not independent of the others - but it is his approach which led to such realisations.

Hilbert now goes on to demonstrate, at least partially, how to build the Theorems of Euclidean geometry from these axioms. He discusses the relationship between Desargues' Theorem and Pappus' Theorem, showing that Desargues Theorem always holds in space but may fail in the plane, that Pappus' Theorem is independent of Desargues', and very nearly[41] proves that Pappus' Theorem implies Desargues'. The fruitfulness of his approach, and the depth to which it will take geometry in the 20th Century, is already apparent.

[39]He does not discuss the consistency, or otherwise, of arithmetic.

[40]As Gray notes ([7] p.255).

[41]As Gray again notes ([7] p.255).

Hilbert also offers a final twist on the "synthetic versus analytic" question. Using his technique with segments, he conjures up new number systems - for example his "desarguian number system". Conversely, he appeals to number systems to prove the consistency of his geometry. Perhaps Darboux was right after all when he wrote "the alliance between geometry and analysis is useful and productive; and that perhaps this alliance is a condition for success to them both." ([3] §I). The closing, definitive, note on this should go to Klein ([10] Note 1):

"*The distinction between modern synthesis and modern analytic geometry must no longer be regarded as essential, inasmuch as both subject-matter and methods of reasoning have gradually taken a similar form in both .... Although the synthetic method has more to do with space-perception and thereby imparts a rare charm to its first simple developments, the realm of space-perception is nevertheless not closed to the analytic method, and the formulae of analytic geometry can be looked upon as a precise and perspicuous statement of geometrical relations.*"

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