Dimension in Transcendental Dynamics
2: The exponential family

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Introduction

In this talk we are interested in transcendental entire functions in the *exponential family*, defined by

$$f_\lambda(z) = \lambda e^z, \quad \text{for } \lambda \in \mathbb{C} \backslash \{0\}.$$

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And even $\dim_H J(f_\lambda) \cap A(f_\lambda) = 2$.

A construction of McMullen

For each $n \in \mathbb{N}$, suppose that $\mathcal{E}_n$ is a finite collection of pairwise disjoint compact subsets of $\mathbb{C}$ such that the following both hold:

(i) If $F \in \mathcal{E}_{n+1}$, then $\exists$ a unique $G \in \mathcal{E}_n$ such that $F \subset G$;

(ii) If $G \in \mathcal{E}_n$, then $\exists$ at least one $F \in \mathcal{E}_{n+1}$ such that $G \supset F$.

We write

$$E_n = \bigcup_{F \subset \mathcal{E}_n} F, \text{ for } n \in \mathbb{N}, \quad \text{and} \quad E = \bigcap_{n \in \mathbb{N}} E_n.$$

A lemma

Lemma 2 (McMullen, 1987)

Suppose that $(\Delta_n)_{n\in\mathbb{N}}$ *and* $(d_n)_{n\in\mathbb{N}}$ *are sequences of positive real numbers, with* $d_n \to 0$ *as* $n \to \infty$*, such that for each* $n \in \mathbb{N}$ *and for each* $F \in \mathcal{E}_n$*, we have*

$$\frac{\text{area}(E_{n+1} \cap F)}{\text{area}(F)} \geq \Delta_n \quad \textit{and} \quad \text{diam}\, F \leq d_n.$$

Then

$$\dim_H E \geq 2 - \limsup_{n\to\infty} \frac{\sum_{k=1}^{n} |\log \Delta_k|}{|\log d_n|}.$$

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- $$\mu(B) \leq \mu(\overline{B}) \leq \frac{\text{Area}(\overline{B})}{\Delta_1\Delta_2\ldots\Delta_k} \leq c' r^\delta \left(\frac{d_k^{2-\delta}}{\Delta_1\Delta_2\ldots\Delta_k}\right).$$

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- *Homework*: Whenever $2 - \delta$ is greater than the lim sup in the statement of the lemma, the quantity in brackets tends to zero as $k \to \infty$, and so, in particular, it is bounded above.
- Thus μ, satisfies an estimate of the required form, so the Hausdorff dimension of E is at least δ.

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- Choose a box $B_0 \subset F$. Set $\mathcal{E}_0 = \{B_0\}$ and, in general,

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- Estimate the diameters, distortion and densities.
- Show that $\dim_H E = 2$.
- *Homework*: Why is $E \subset J(f_\lambda) \cap A(f_\lambda)$?

Related results

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  the densities tend to one so quickly that $J(f) \cap A(f)$ has positive area.
- S. showed that the same is true for, for example, functions of the form

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- It is known (Rempe/Rippon/Stallard) that $J(f_\lambda) \setminus End_\lambda \subset A(f_\lambda)$ whereas (Rempe) End_λ contains points which escape at every possible rate.
- Mayer showed that $End_\lambda \cup \{\infty\}$ is connected but End_λ is totally disconnected. It follows that End_λ has topological dimension one.

Proof (sketch)

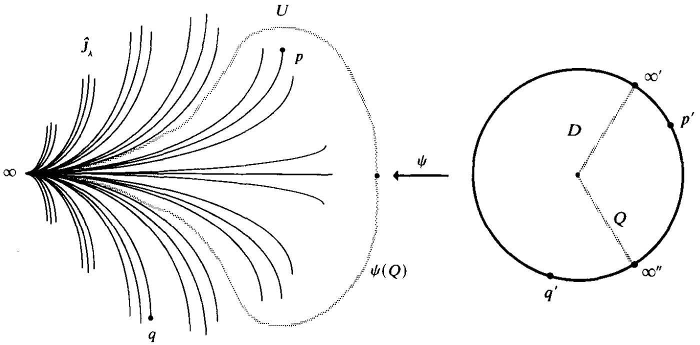

Image: Mayer (1990)

Karpińska's paradox

Theorem 3

Suppose that $\lambda \in (0, e^{-1})$*. Then*

$$\dim_H End_\lambda = 2$$

but

$$\dim_H J(f_\lambda) \backslash End_\lambda = 1.$$

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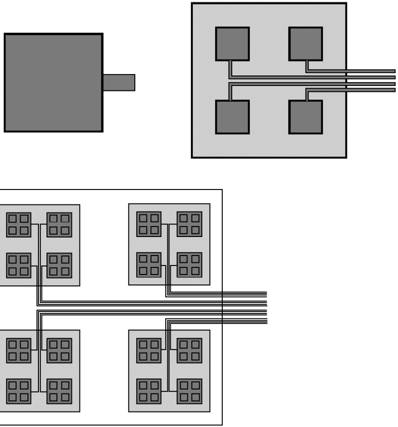

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For $p \in (1, \infty)$ and ξ large, $\dim_H I_{p,\xi} \leq 1 + 1/p$.

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Lemma 5

For $p \in (1, \infty), \xi > 0$, we have $J(f_\lambda) \backslash End_\lambda \subset I_{p,\xi}$.

Annular itineraries

- Define the *Slow escaping set* $L(f)$ as

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- Define the *Uniformly slowly escaping set* $L_U(f)$ as

$$\{z : \exists N,\ 1 < R,\ 0 < C_1 < C_2 \text{ s.t. } C_1 R^n \leq |f^n(z)| \leq C_2 R^n, n \geq N\}$$

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Theorem 6 (S. 2014)

$\dim_H L_U(f_\lambda) = 1.$

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- Use 'countable stability' (again) and a result regarding inverse images to extend this result to the set

$$\bigcup_{n \in \mathbb{Z}} f^n(L').$$

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