Dimension in Transcendental Dynamics
3: The dimension of the Julia set

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Proposition 4

$\dim_H J(f) = 2$ *is possible.*

A first step

Theorem 1 (Stallard, 1991)

Given $\delta > 0$ there is a transcendental entire function f such that

$$1 \leq \dim_H J(f) \leq 1 + \delta.$$

Definition of the function

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- Finally set $f_K(z) = E(z) - K$, for $K > 0$.

Properties of the function

- Show that, for large z

$$E(z) \sim -1/z, \quad \text{for } z \in \mathbb{C} \backslash \overline{G}$$

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- Recalling that $f_K(z) = E(z) - K$, for $K > 0$, we choose K so large that

$$\mathbb{C} \backslash \overline{G} \subset F(f_K).$$

Sketch of proof

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- HOMEWORK: It is enough to show that

$$\sum_{F \in \mathcal{E}_{n+1}} (\text{diam } F)^{1+\delta} \leq \sum_{F \in \mathcal{E}_n} (\text{diam } F)^{1+\delta}, \quad \text{for large } n.$$

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- Or indeed to show that, for all sufficiently large n, we have for each $G \in \mathcal{E}_n$ that

$$\sum_{\substack{F \in \mathcal{E}_{n+1}, \\ F \cap G \neq \emptyset}} \left(\frac{\text{diam } F}{\text{diam } G} \right)^{1+\delta} \leq 1.$$

Two half steps

Theorem 2 (Stallard, 1997, 2000)

For each $p \in (1, 2]$ there is a transcendental entire function f such that

$$\dim_H J(f) = p.$$

Definition of the function

- Let $p \in (0, \infty)$ and

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- Finally set $f_{p,K}(z) = E_p(z) - K$, for $K > 0$.
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- Finally set $f_{p,K}(z) = E_p(z) - K$, for $K > 0$.
- If $K = K(p)$ is sufficiently large, then
$$\dim_H J(f_{p,K}) = 1 + \frac{1}{1+p}.$$
- Proof omitted (not even homework).

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Theorem 3 (Stallard, 1996)

If $f \in \mathcal{B}$, then $\dim_H J(f) > 1$.

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- This gives a conformal iterated function system, formed by strictly contracting, conformal maps $G_p : Q \to Q$.
- Such a system has a unique maximal compact invariant set J_B.

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- The result follows since $J_B \subset J(F)$.

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- ‘In our example, $J(f)$, $J(f) \setminus A(f)$ and $(J(f) \cap I(f)) \setminus A(f)$ are each as small as is possible for a transcendental entire function; in some sense, our example is the "least chaotic" or "most normal" transcendental entire function’.

Epilogue

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Theorem 5 (Stallard, 1994)

Let f be a transcendental meromorphic function which has at least one pole and is not conjugate to a self-map of the punctured plane. Then the Hausdorff dimension of the Julia set $J(f)$ satisfies

$$0 < dim_H J(f) \leq 2,$$

and the bounds are sharp.

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