A new characterisation of the Eremenko-Lyubich class

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Example

Let $f(z) = \exp(z^2)$. This has a critical value equal to 1 and a finite asymptotic value equal to 0.

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Example

$f(z) = (\sin z)/z$.
This function has an asymptotic value equal to zero, and an infinite set of critical values all within $\{w : |w| \leq 1\}$.

Some results on the Eremenko-Lyubich class

Functions in this class have been much investigated, particularly in complex dynamics.

Theorem (Eremenko and Lyubich 1992)

If $f \in \mathcal{B}$, then f has no escaping Fatou components.

Theorem (Stallard 1996)
(See also Barański, Karpińska and Zdunik 2009)

If $f \in \mathcal{B}$, then $\dim_H J(f) > 1$.

Theorem (Rottenfusser,Rückert,Rempe and Schleicher 2011)

There exists $f \in \mathcal{B}$ such that every path-connected component of $J(f)$ is bounded.

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Lemma (Eremenko and Lyubich 1992)

If $f \in \mathcal{B}$, then there is a constant $R_0 > 0$ such that

$$\left| z \frac{f'(z)}{f(z)} \right| \geq \frac{1}{4\pi}(\log |f(z)| - \log R_0), \quad \textit{for } z \in D_{R_0}.$$

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Theorem (S. 2011)

Suppose that f is a transcendental entire function. Then, either $\eta_f = \infty$ and $f \in \mathcal{B}$, or $\eta_f = 0$ and $f \notin \mathcal{B}$.

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- We use this classification to complete the proof.

A classification of transcendental singularities

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- $f(z) = \exp z$ has a direct logarithmic singularity over zero.
- Otherwise it is called *non-logarithmic*.
- $$f(z) = \exp\left(\sum_{k=1}^{\infty} \left(\frac{z}{2^k}\right)^{2^k}\right)$$

  has a direct non-logarithmic singularity over zero.

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The set of projections of direct non-logarithmic singularities of f is bounded.

Proof of lemma – the set of projections of direct logarithmic singularities of f is bounded.

Theorem

Suppose that W is a domain, and $g : W \to \mathbb{D}^*$ is a covering map. Then exactly one of the following holds:

(i) *there exists a conformal map $\phi : W \to \mathbb{H}$ such that $g = \exp \circ \phi$;*

(ii) *there exists a conformal map $\phi : W \to \mathbb{D}^*$ such that $g = (\phi)^m$, for some $m \in \mathbb{N}$.*

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Corollary

Suppose that f is a transcendental entire function with a logarithmic singularity over a point $a \in \widehat{\mathbb{C}}$. Then there exist a neighbourhood of the singularity, $W = U(r)$, and conformal maps $\phi : W \to \mathbb{H}$ and $h : B(a, r) \backslash \{a\} \to \mathbb{D}^*$ such that $h \circ f = \exp \circ \phi$.

Proof of lemma – the set of projections of indirect logarithmic singularities of f is bounded.

Theorem (Bergweiler and Eremenko 1995)

Suppose that f is a transcendental entire function with an indirect singularity with projection $a \in \widehat{\mathbb{C}}$. Suppose that a is not the limit of critical values of f. Then there exists a sequence of asymptotic values (a_n), which converge to a, a sequence of disjoint unbounded simply connected domains (U_n) such that $D_n = f(U_n)$ is a disc with $a_n \in \partial D_n$, and a sequence of asymptotic curves (Γ_n) such that $\Gamma_n \subset U_n$, $f(\Gamma_n)$ is a radius of D_n ending at a_n, and f is univalent in U_n.

Proof of lemma – the set of projections of direct non-logarithmic singularities of f is bounded.

Theorem (S. 2011)

Suppose that f is a transcendental entire function, with a direct non-logarithmic singularity with projection $a \in \widehat{\mathbb{C}}$. Then at least one of the following holds:

(i) *a is the limit of critical values of f;*

(ii) *every neighbourhood of this singularity contains a neighbourhood of another singularity that is either indirect or logarithmic, and whose projection is different from a.*

Proof of lemma – the set of projections of direct non-logarithmic singularities of f is bounded.

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Theorem (Bergweiler and Eremenko 2008)

Suppose that f is a transcendental entire function, with a direct non-logarithmic singularity with projection $a \in \mathbb{C}$. Then every neighbourhood of this singularity is also a neighbourhood of other direct singularities with projection a.