# Cryptographic applications of very large cardinals

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2017

# Topics

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Algebra-Self-distributive algebras

Set theory-Rank-into-rank cardinals

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# Laver-like algebras

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If $(X, *)$ is a Laver-like algebra, then define the Fibonacci terms $t_n$ for $n \geq 1$ by letting $t_1(x, y) = y, t_2(x, y) = x$, and $t_{n+2}(x, y) = t_{n+1}(x, y) * t_n(x, y)$. Then for all $x, y$ there is some $n$ where $t_n(x, y) \in \mathrm{Li}(X)$.

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# Large cardinals

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- Rank-into-rank, $j : V_\lambda \to V_\lambda$-Laver-like algebras $\mathcal{E}_\lambda$.
- $n$-huge-Laver-like algebras.
- Extendible-Laver-like algebras.
- Supercompact
- Woodin
- Measurable
- Ramsey
- Erdos
- Indescribable
- Weakly compact
- Mahlo
- Inaccessible

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Let $\mathcal{E}_\lambda$ be the set of all elementary embeddings $j : V_\lambda \to V_\lambda$. Then define an operation $*$ on $\mathcal{E}_\lambda$ by letting $j * k = \bigcup_{\alpha<\lambda} j(k|V_\alpha)$.
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$(\mathcal{E}_\lambda, *)$ is self-distributive.

If $\gamma$ is a limit ordinal with $\gamma < \lambda$, then define a congruence $\equiv^\gamma$ on $(\mathcal{E}_\lambda, *)$ by letting $j \equiv^\gamma k$ iff $j(x) \cap V_\gamma = k(x) \cap V_\gamma$ for all $x \in V_\gamma$.
Then $(\mathcal{E}_\lambda / \equiv^\gamma, *)$ is a Laver-like algebra.

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(ZFC) If $n$ is a natural number, then there is a unique algebra $A_n = (\{1, \ldots, 2^n\}, *)$ such that

1. $x * (y * z) = (x * y) * (x * z)$, and
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If $(X, t)$ is an $n+1$-ary self-distributive algebra, then define the hull $\Gamma(X, t) = (X^n, *)$ where $*$ is the binary operation defined by $(x_1, \ldots, x_n) * (y_1, \ldots, y_n) = (t(x_1, \ldots, x_n, y_1), \ldots, t(x_1, \ldots, x_n, y_n))$.

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If $t : X^{n+1} \to X$, then for $a_1, \ldots, a_n \in X$, define $L_{t,a_1,\ldots,a_n} : X \to X$ by letting $L_{t,a_1,\ldots,a_n}(x) = t(a_1, \ldots, a_n, x)$. Then $t$ is self-distributive iff each $L_{t,a_1,\ldots,a_n}$ is an endomorphism of $(X, t)$.

If $(X, t)$ is an $n+1$-ary self-distributive algebra, then define the hull $\Gamma(X, t) = (X^n, *)$ where $*$ is the binary operation defined by $(x_1, \ldots, x_n) * (y_1, \ldots, y_n) = (t(x_1, \ldots, x_n, y_1), \ldots, t(x_1, \ldots, x_n, y_n))$. Then $\Gamma(X, t)$ is a self-distributive algebra. We say that an $n+1$-ary self-distributive algebra $(X, t)$ is Laver-like if its hull $\Gamma(X, t)$ is Laver-like.

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The elements in $\Diamond(X, t^\bullet)$ are factorization trees of elements in the algebra $(X, t^\bullet)$.

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**Ko-Lee key exchange**

If $X$ is a functionally endomorphic Laver table, then $(\Gamma(X) \setminus \mathrm{Li}((\Gamma(X)), \circ)$ may be used as a platform for the Ko-Lee key exchange.

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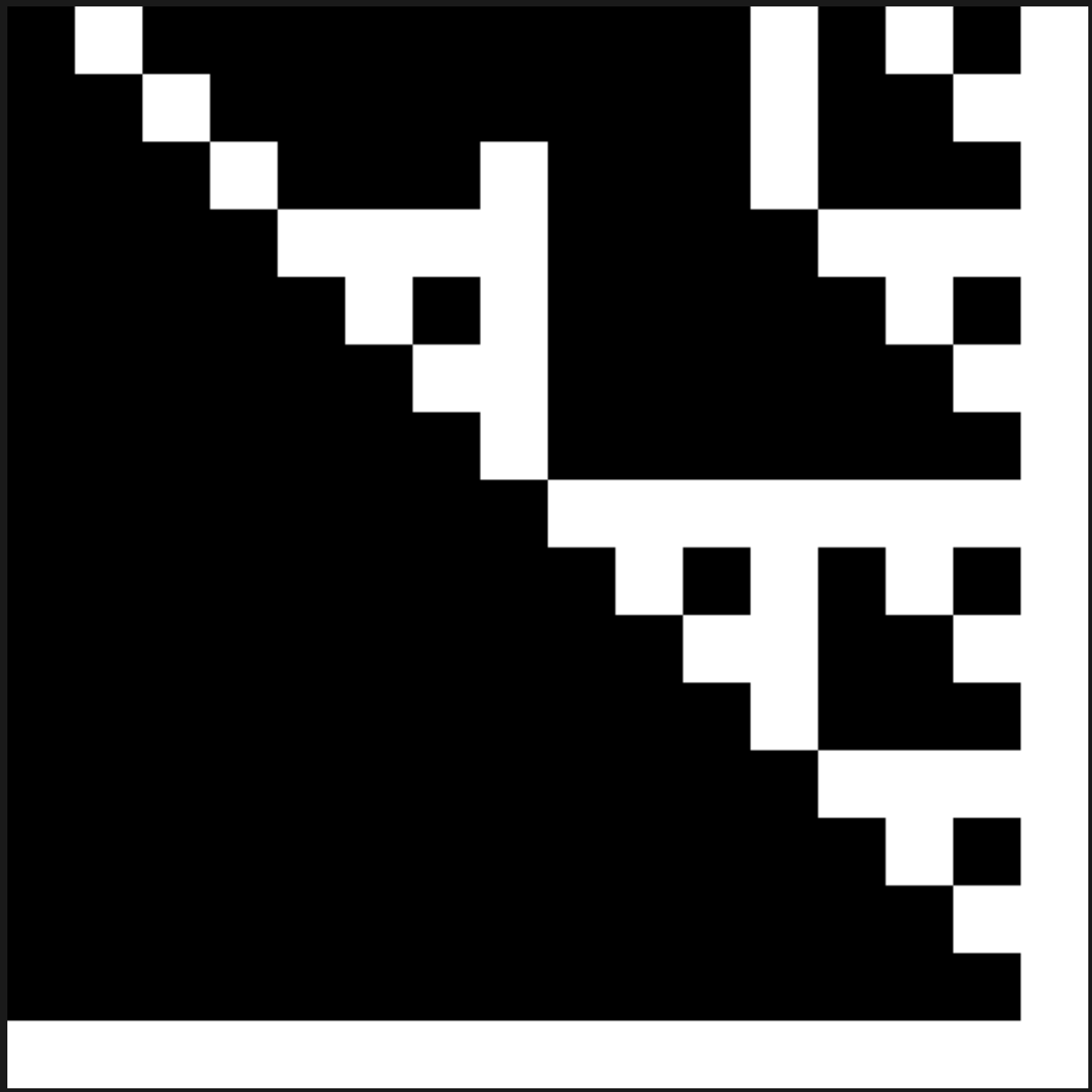
# Pictures: $A_4$



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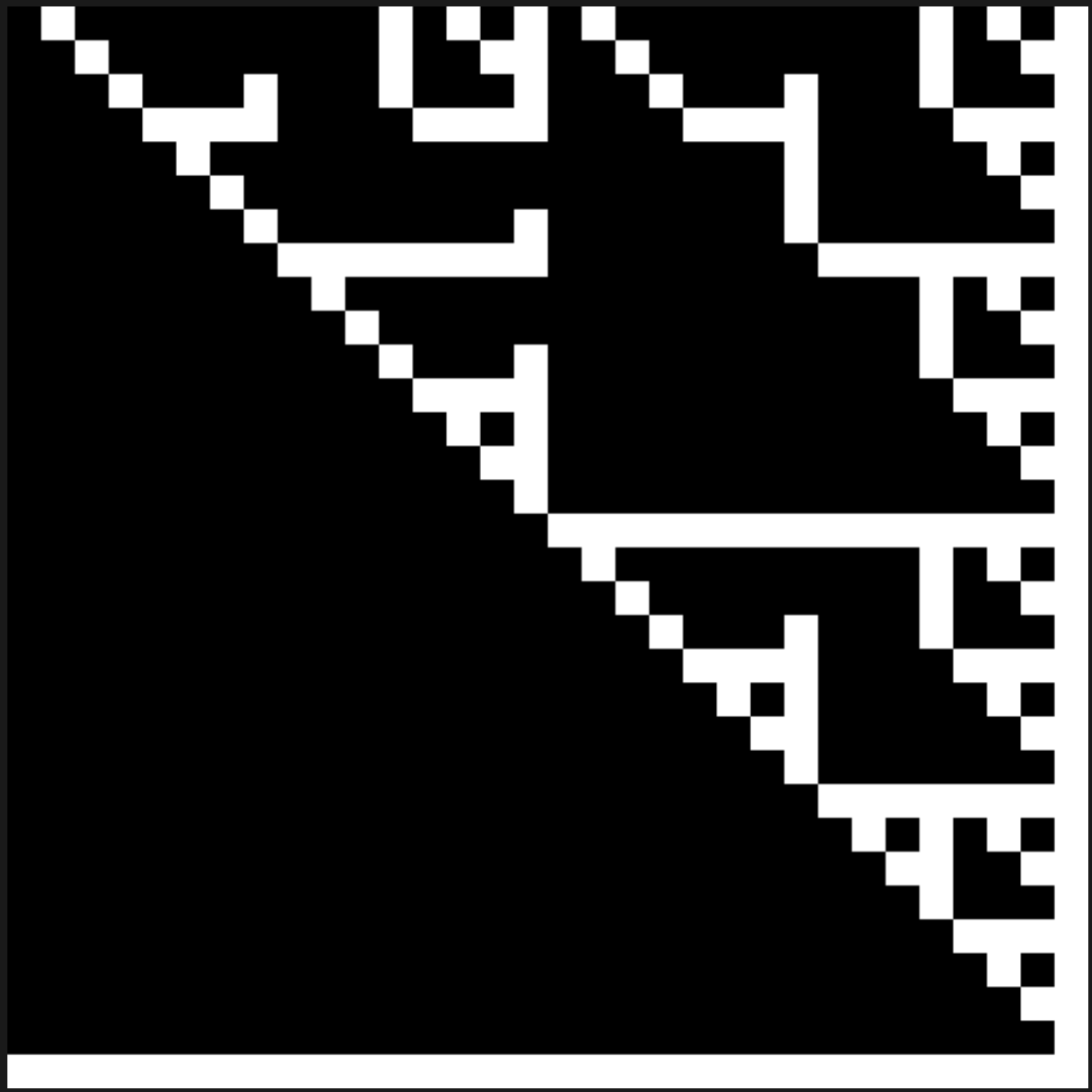

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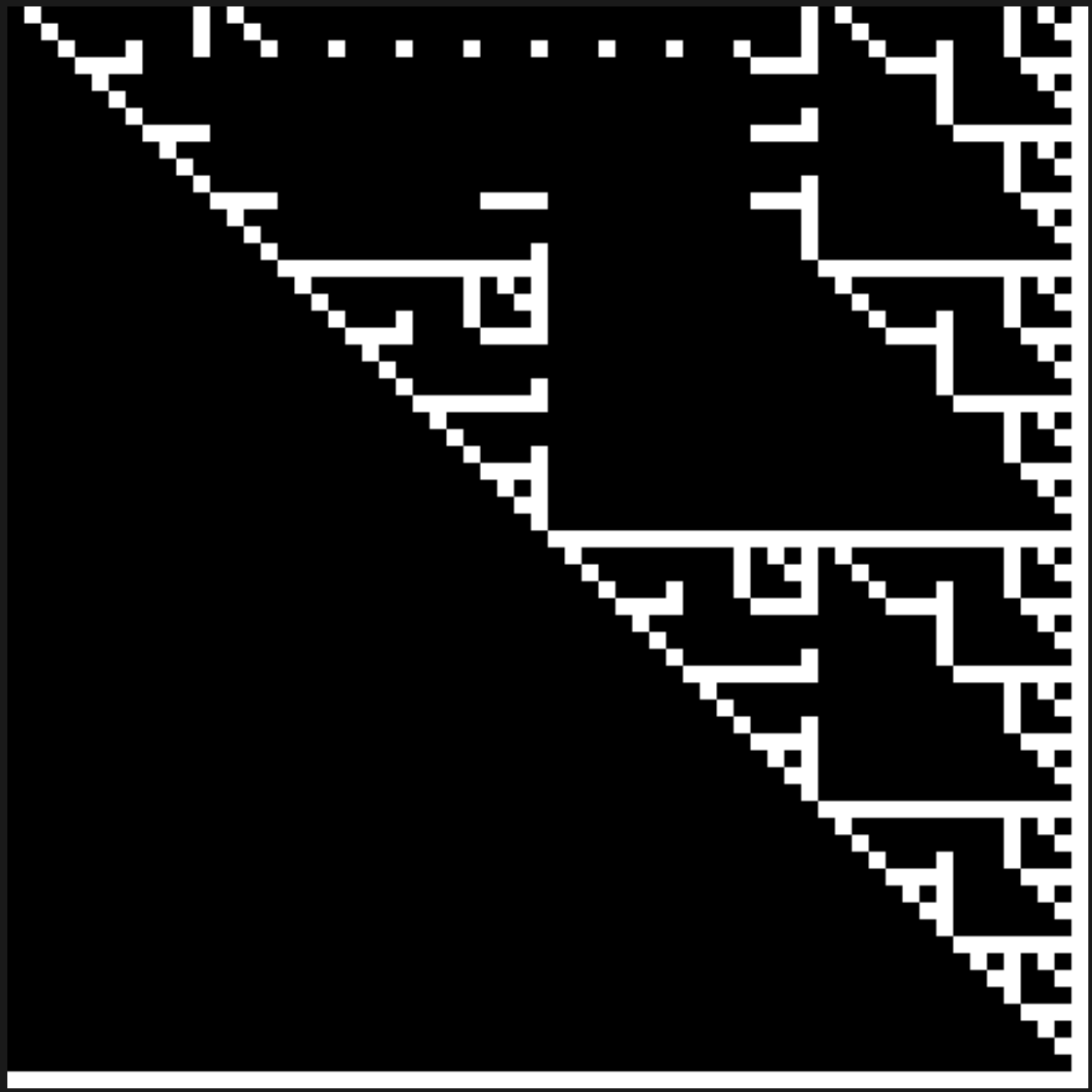

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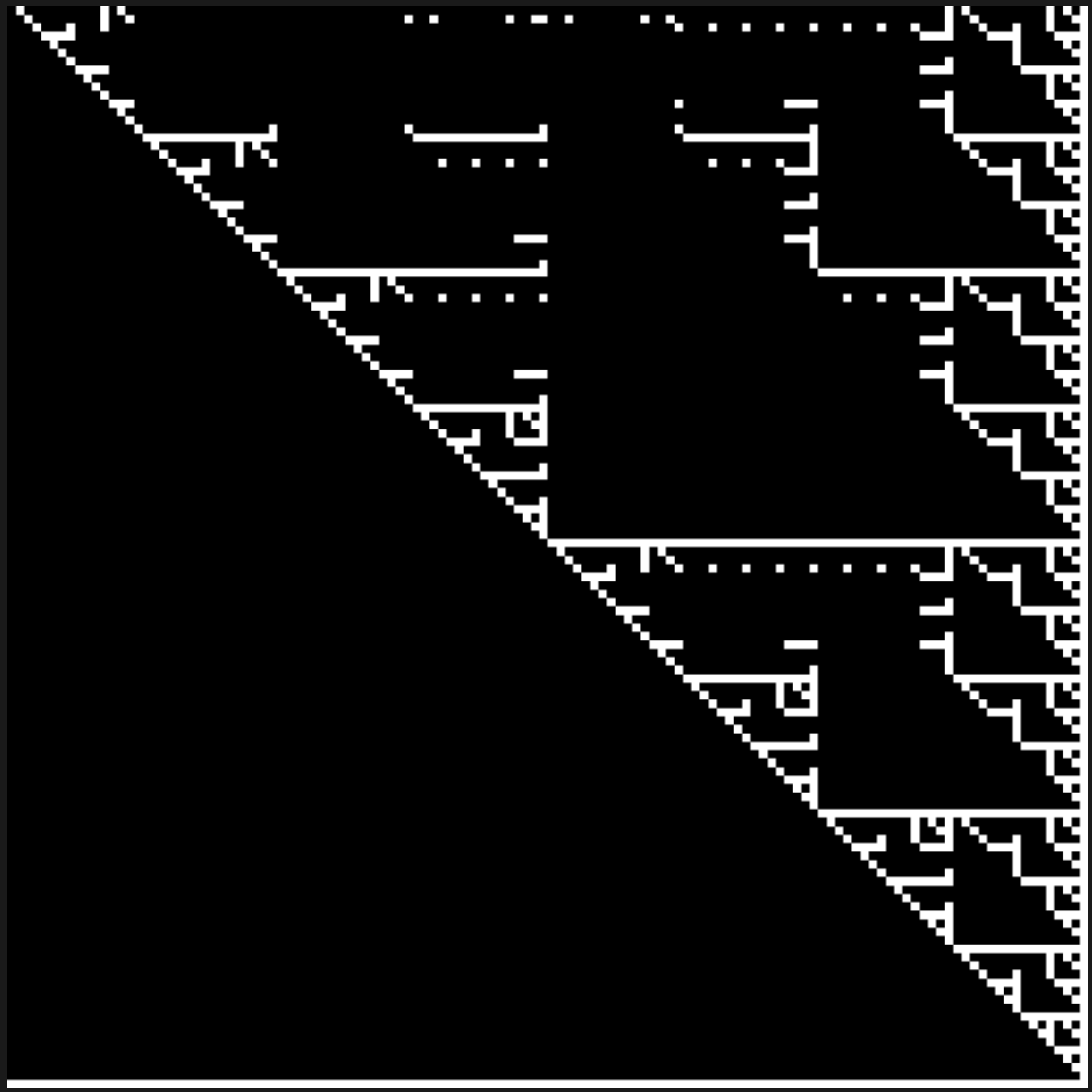

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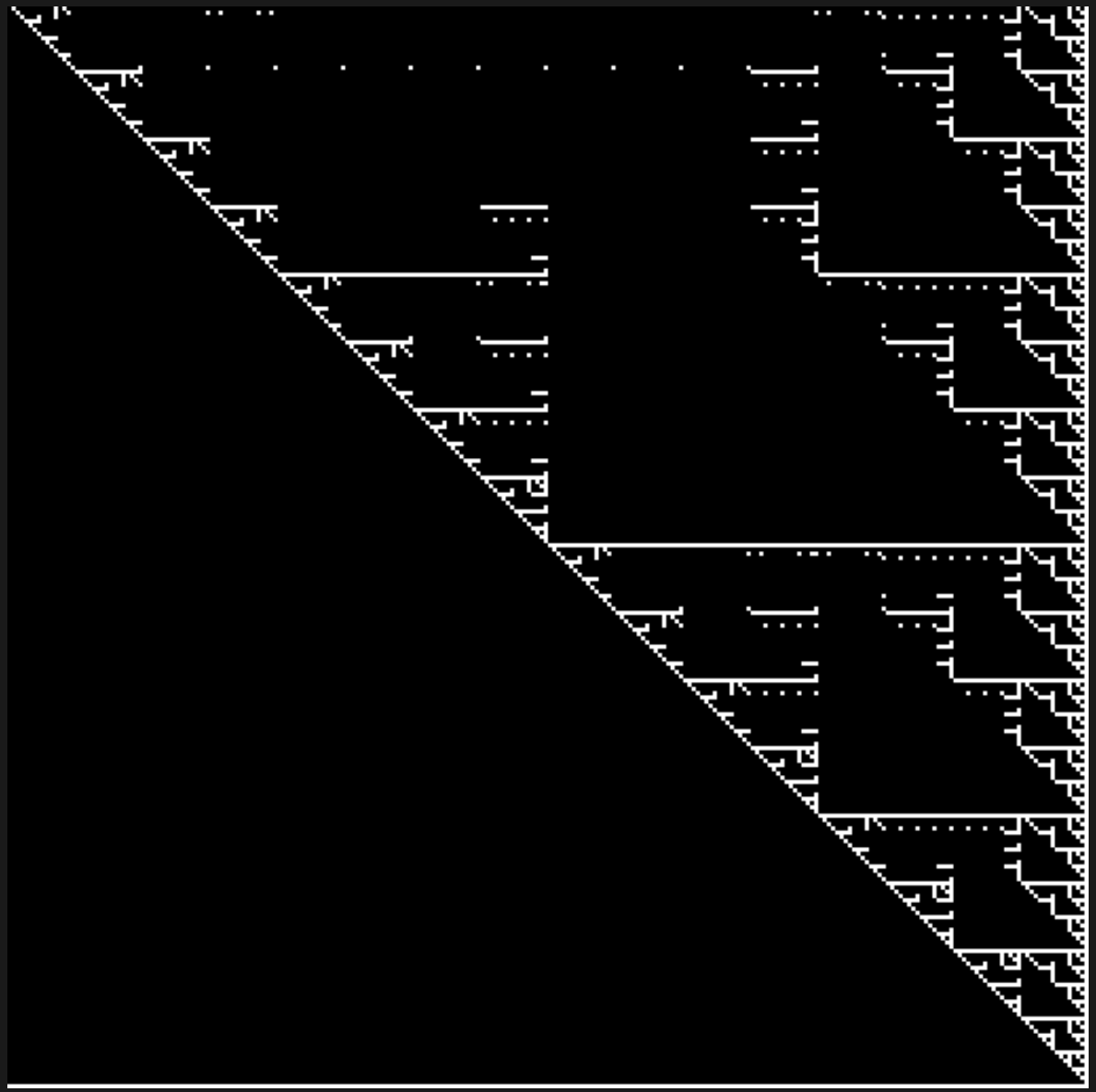

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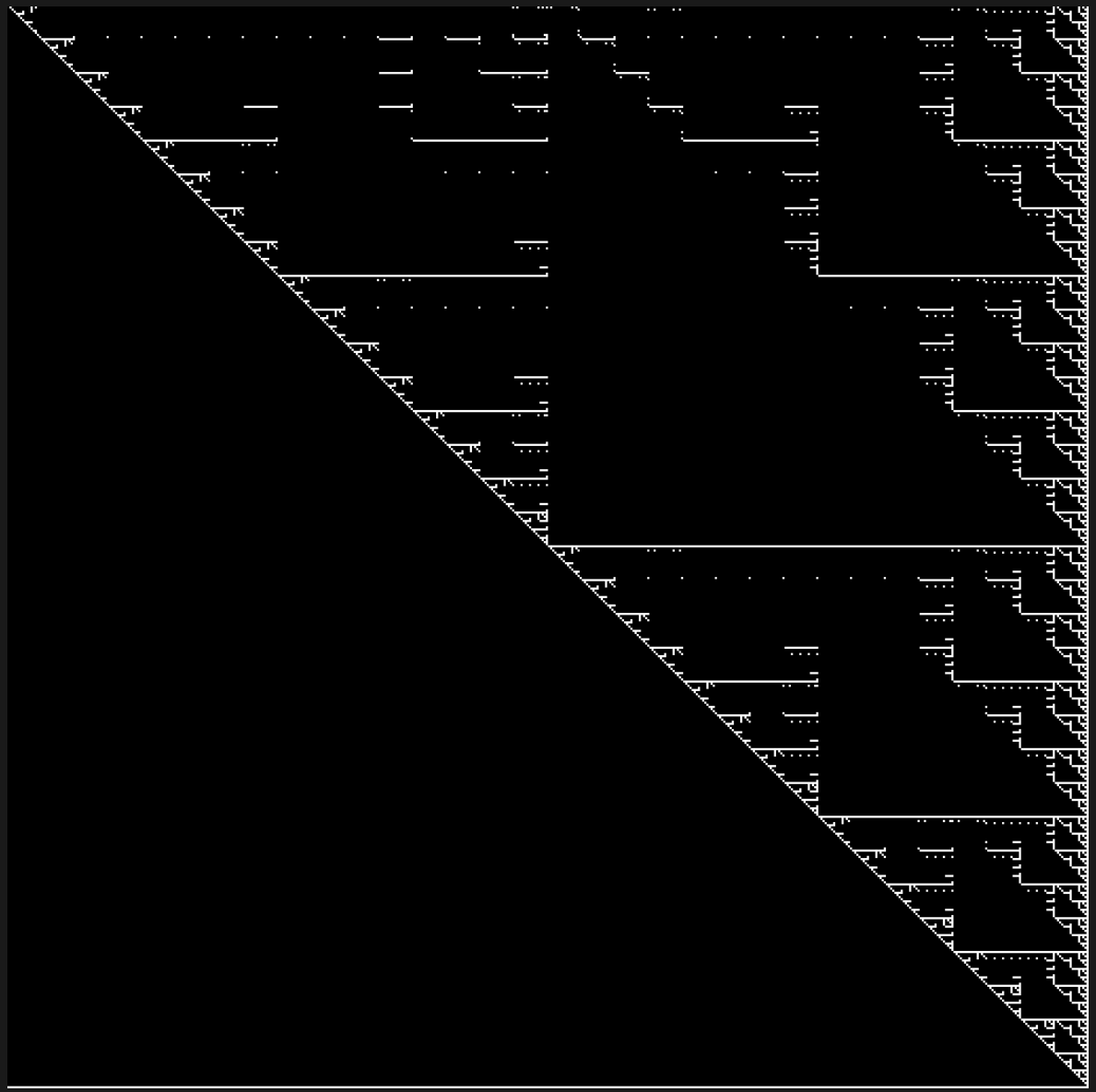

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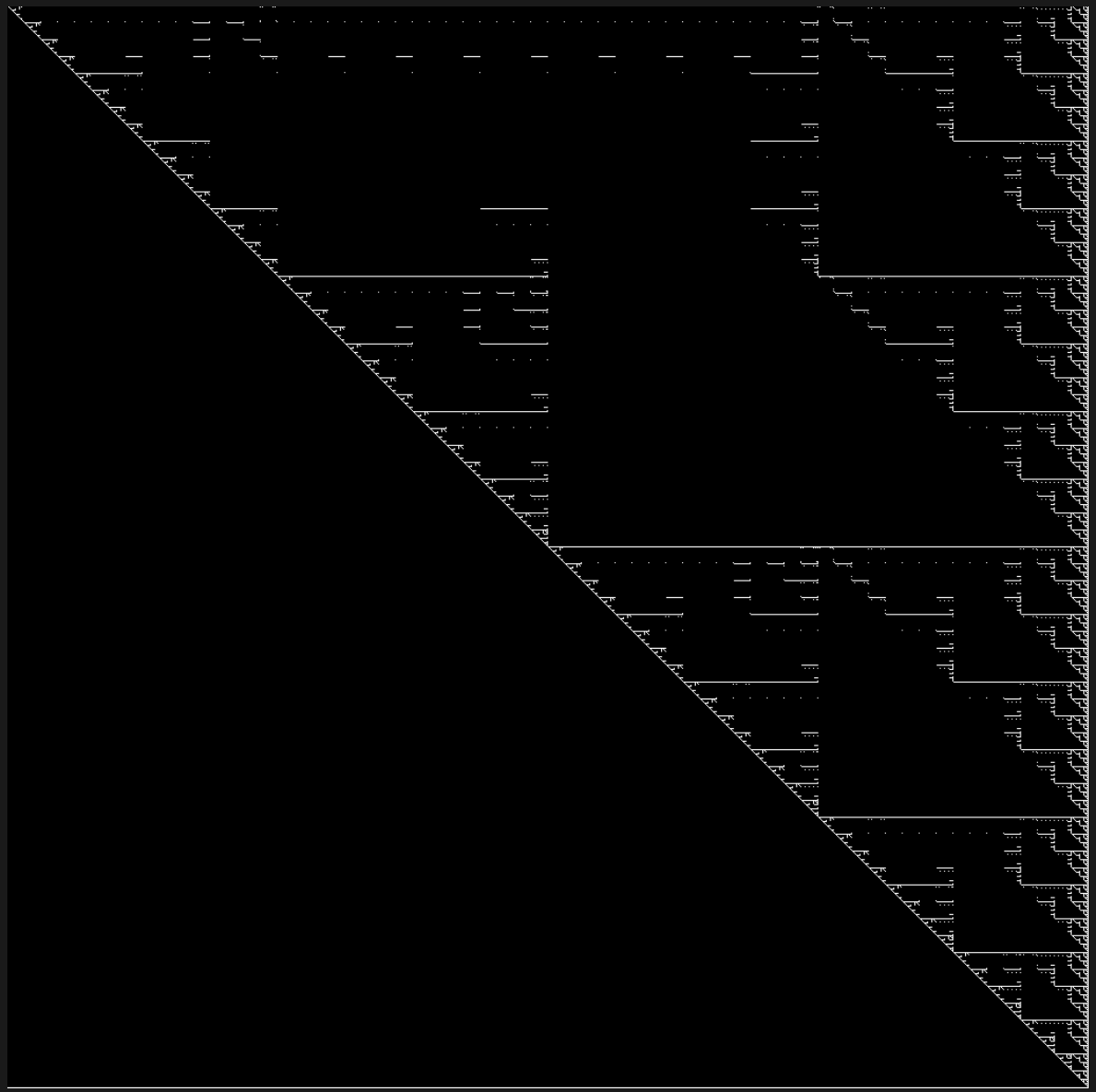

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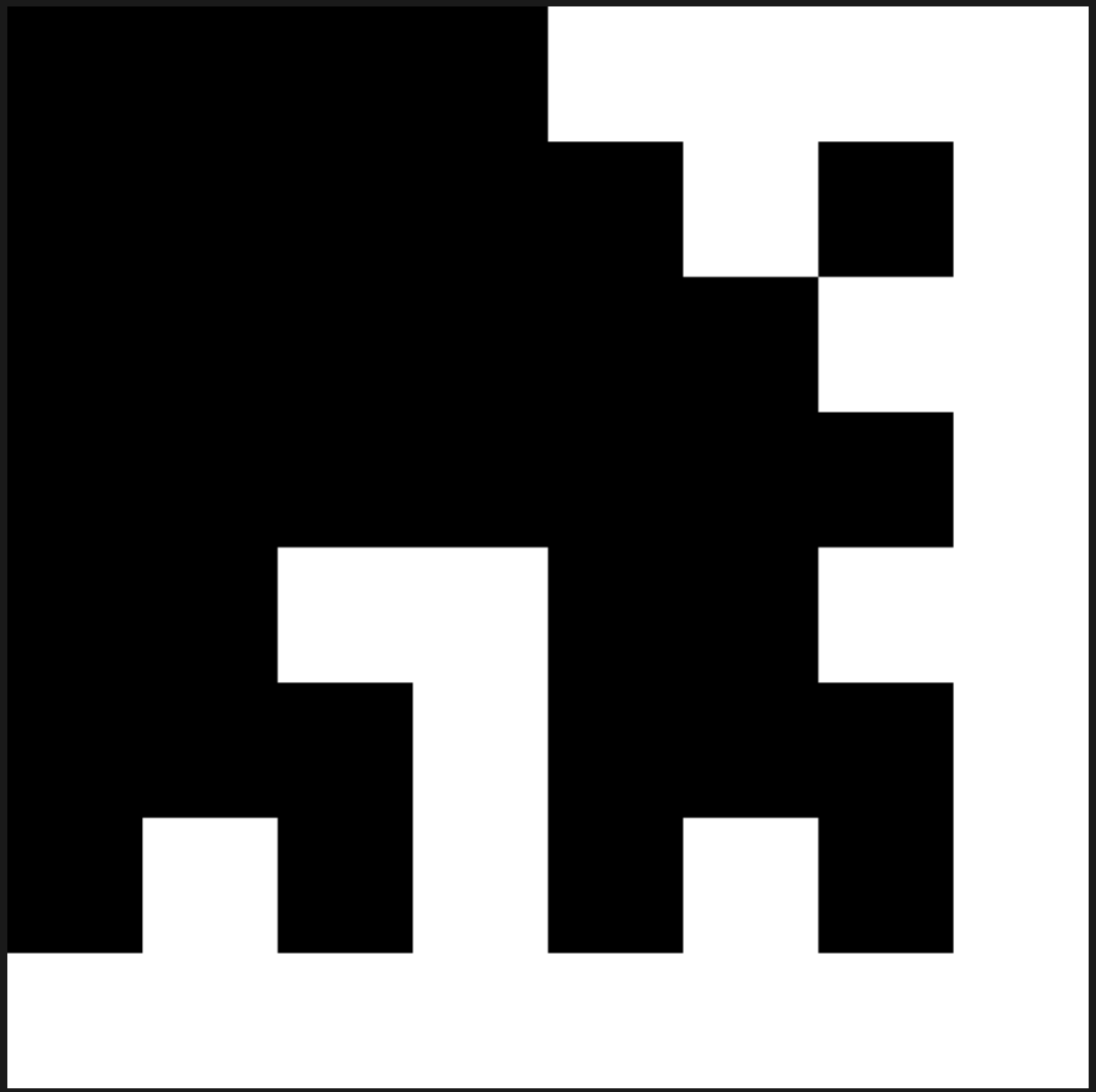

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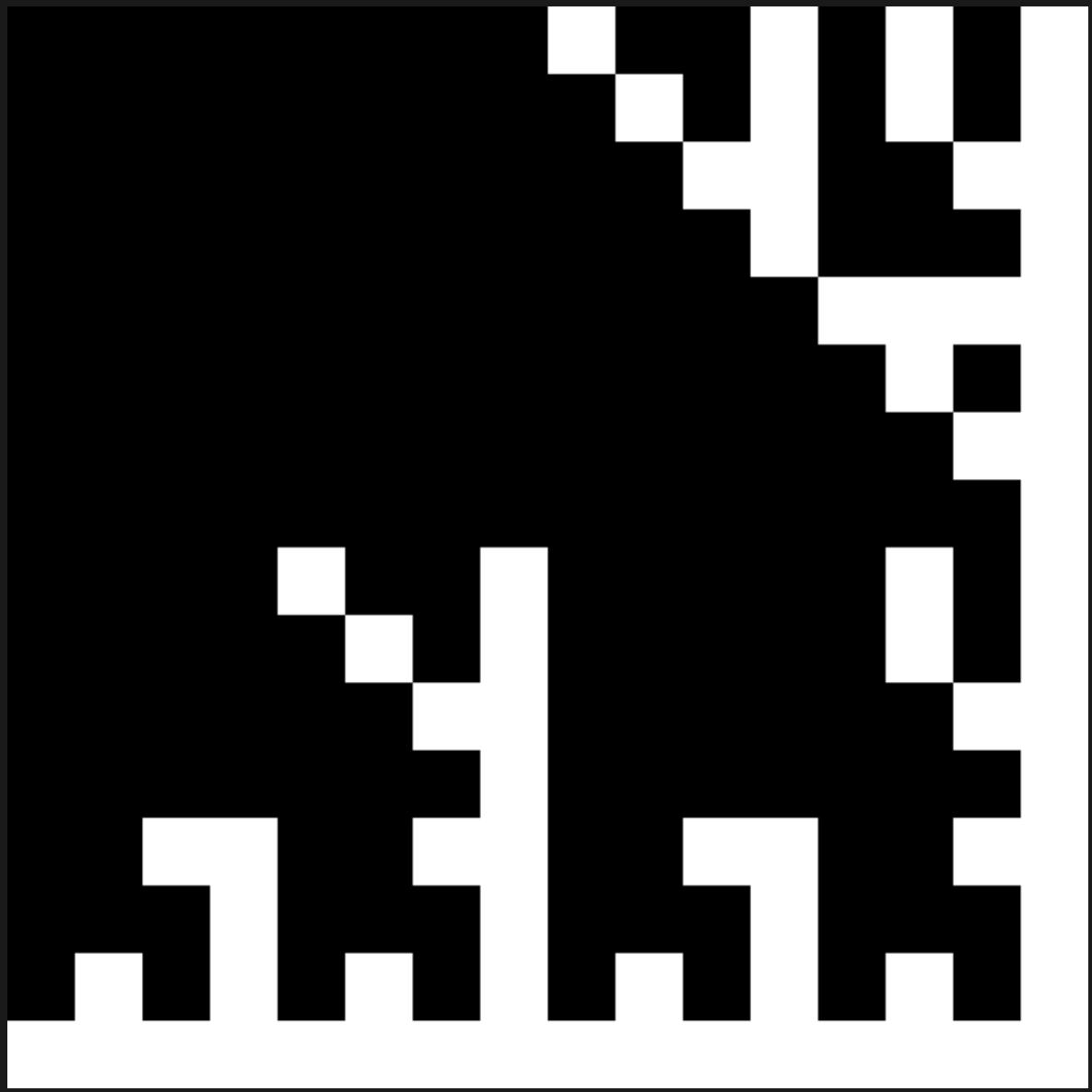

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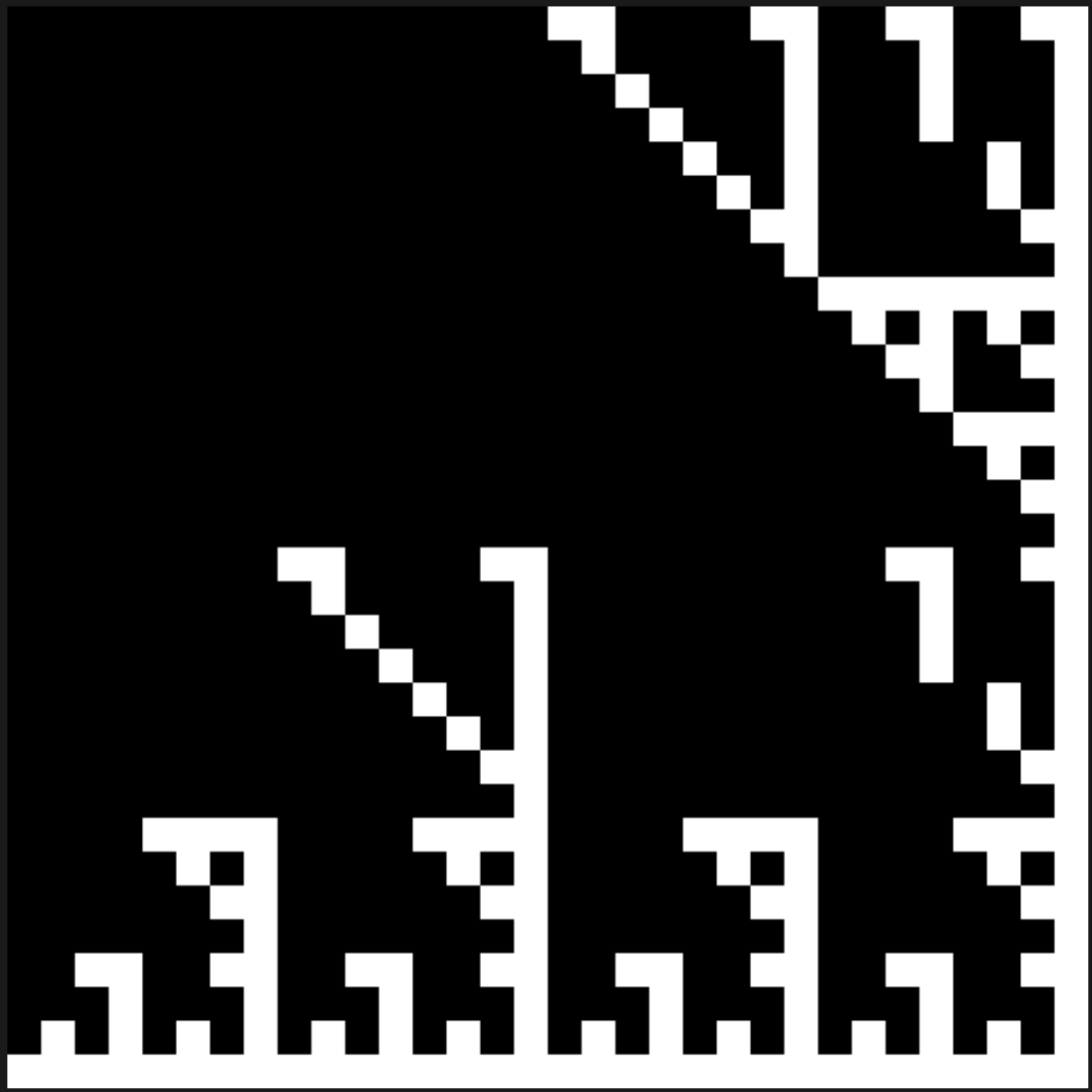

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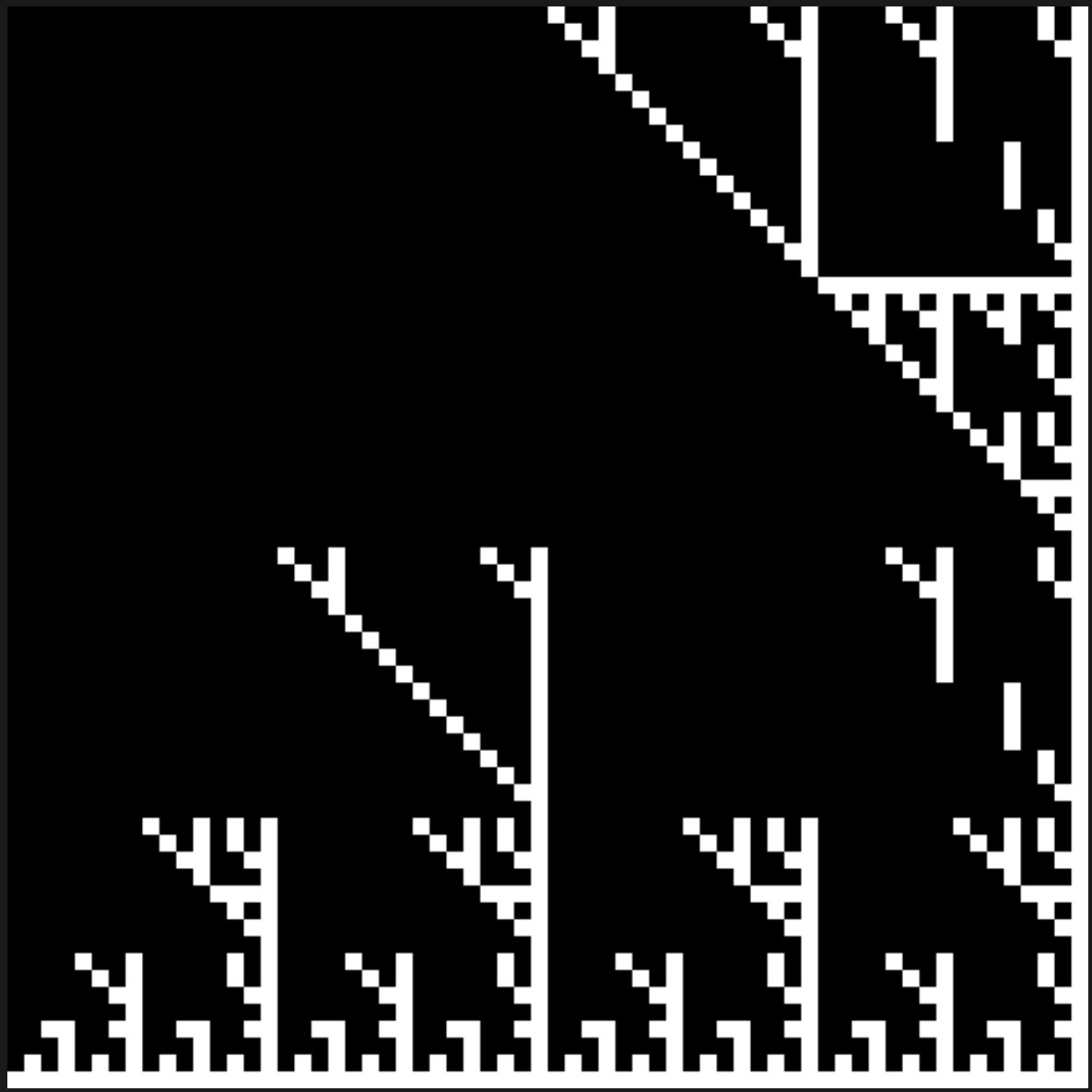

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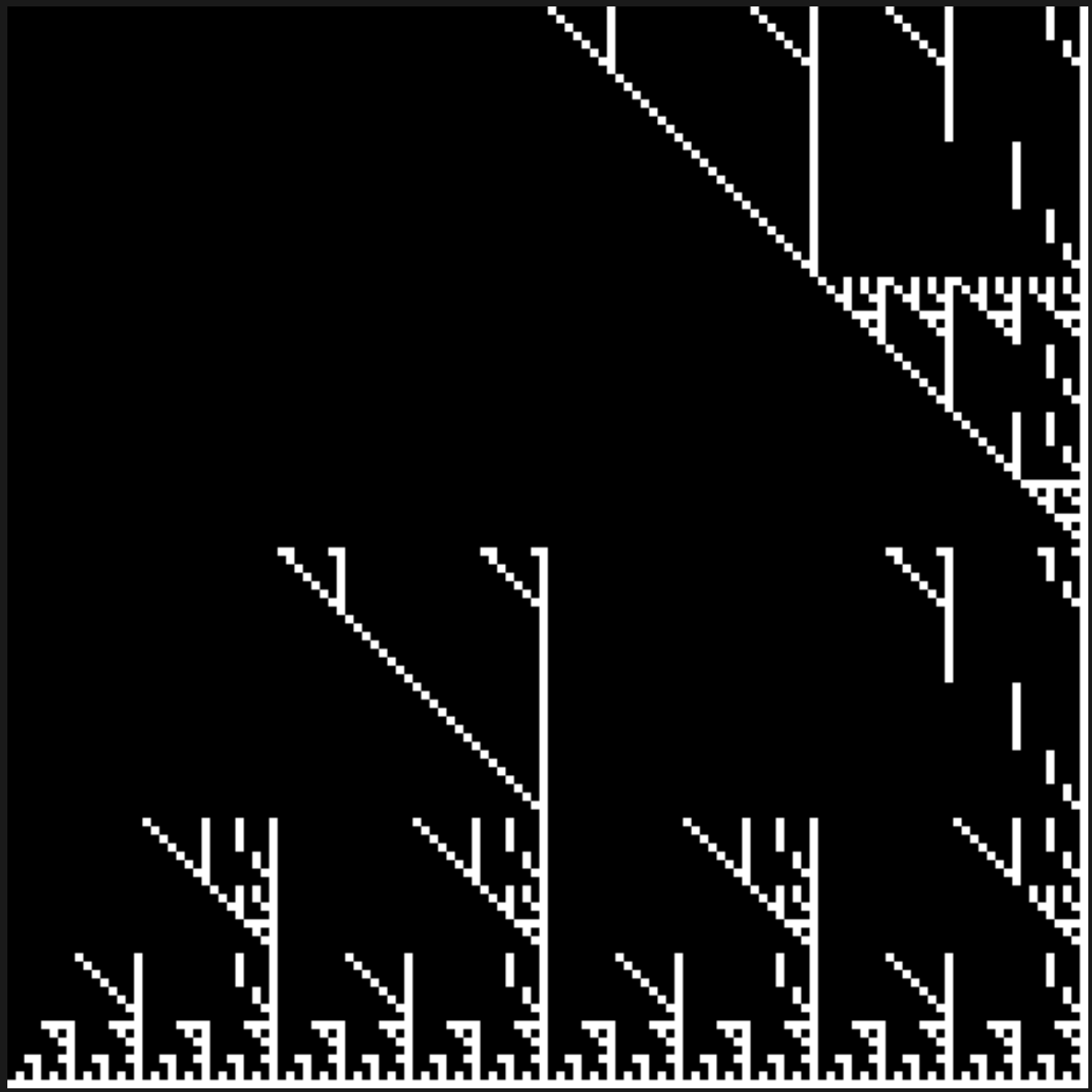

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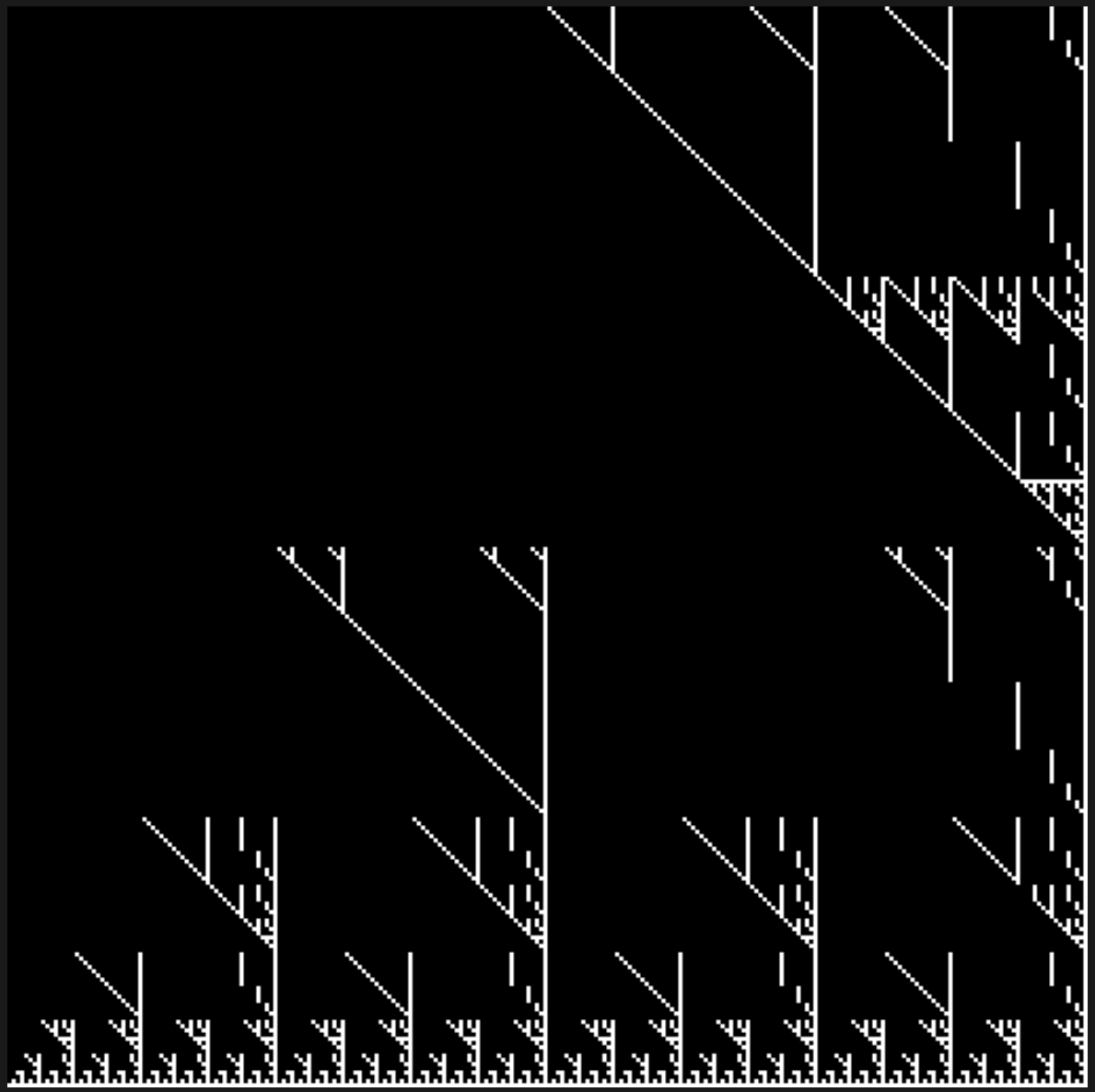

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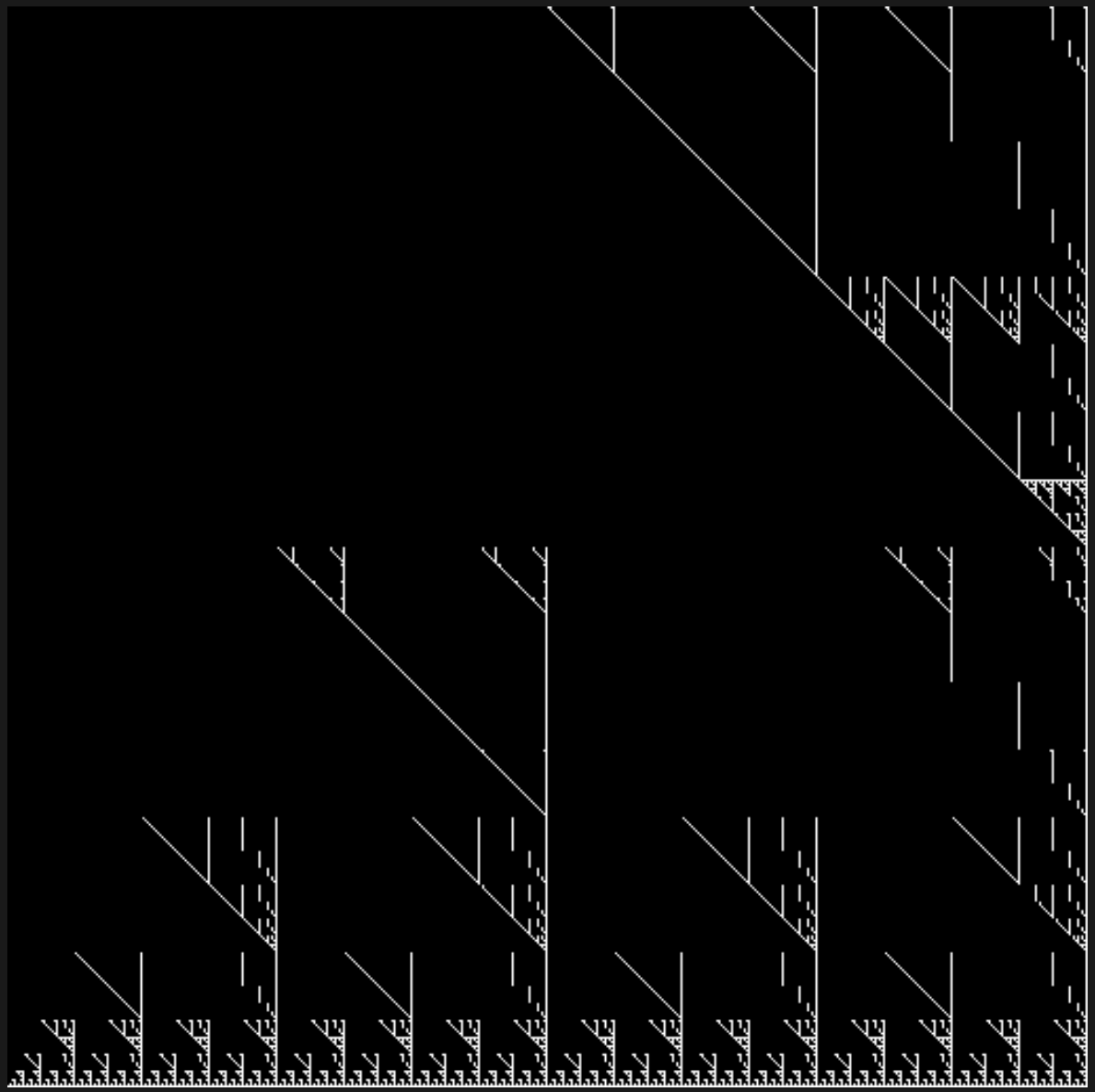

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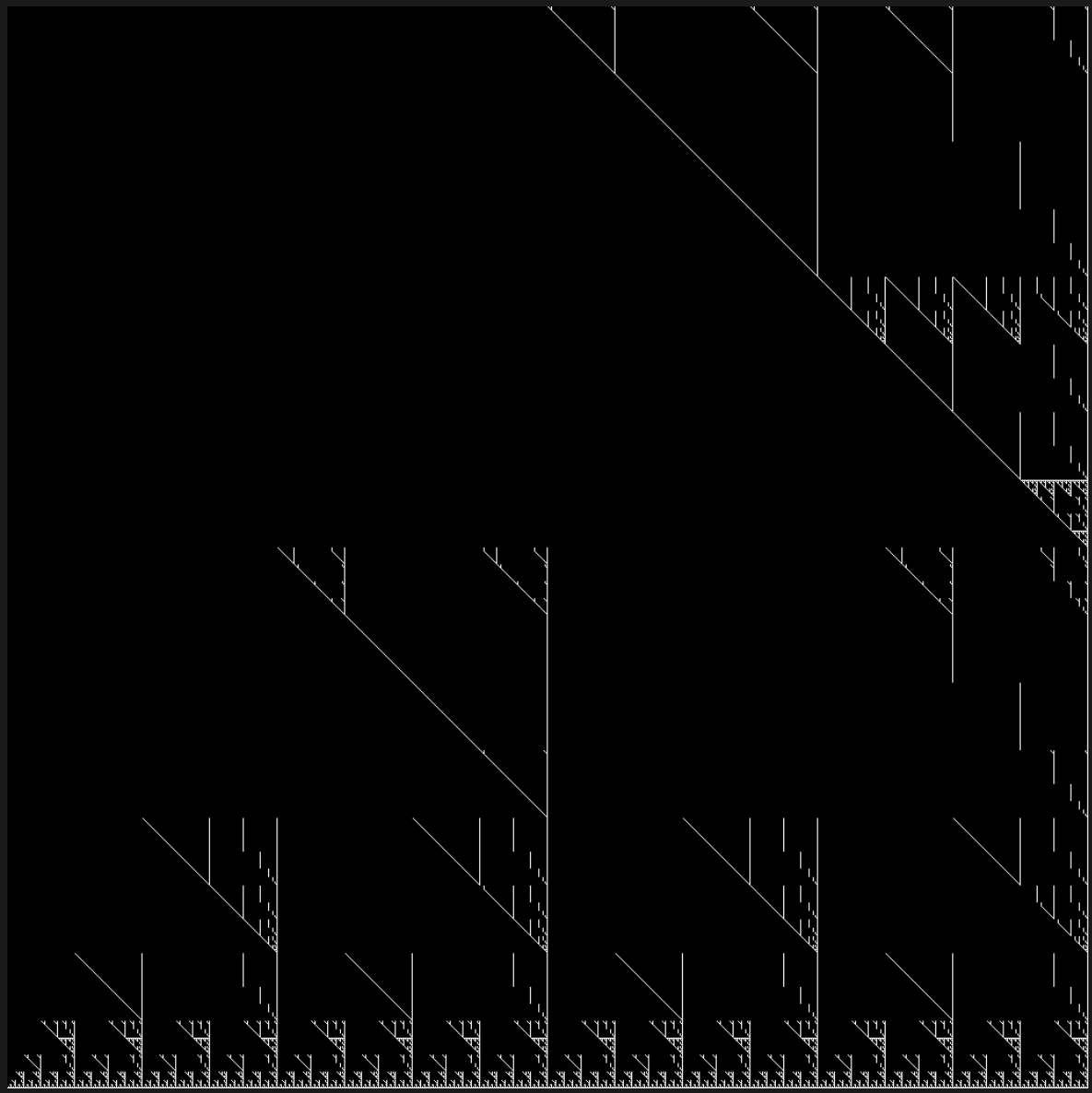
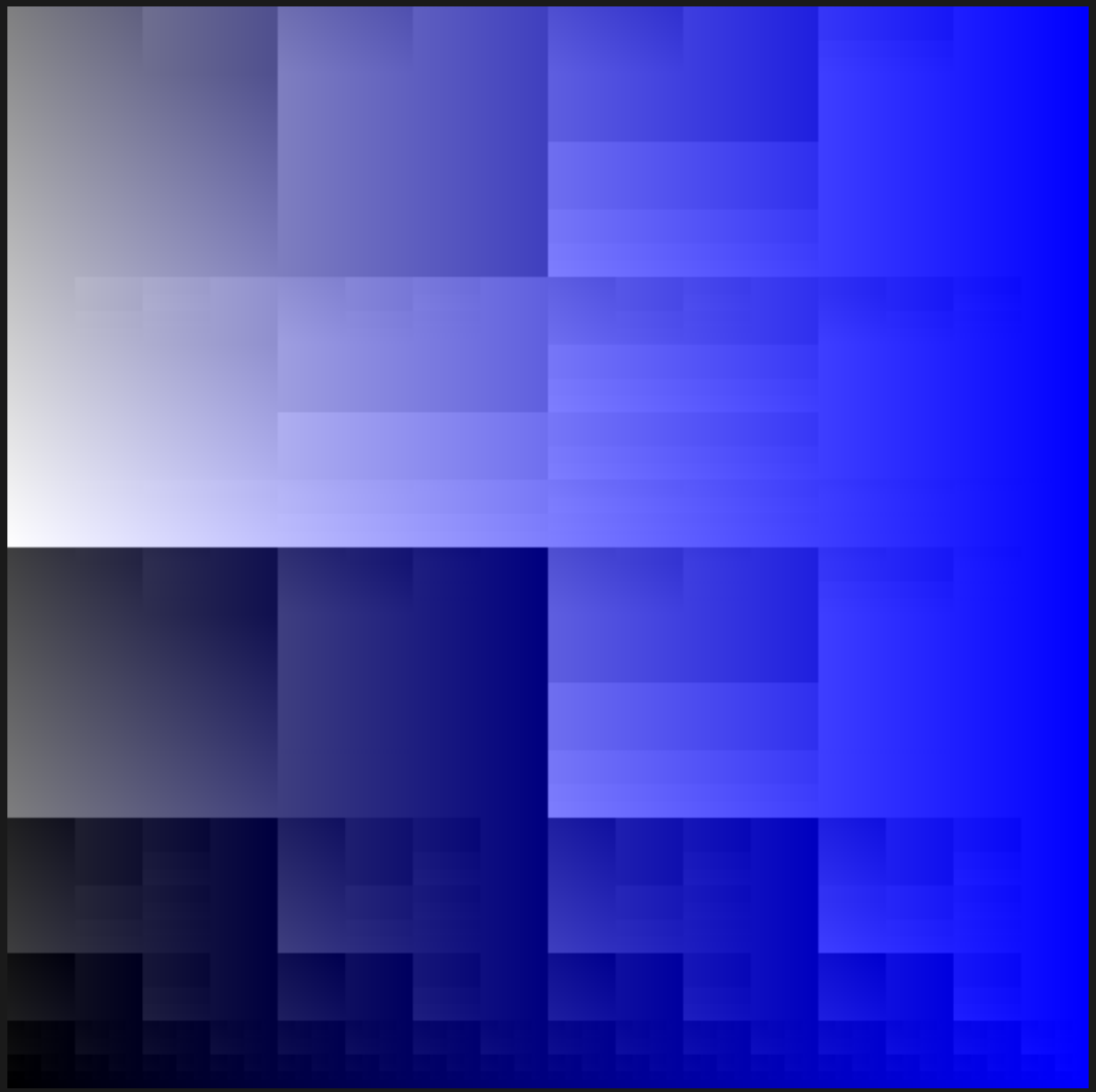


# Pictures:Classical Laver table heat map



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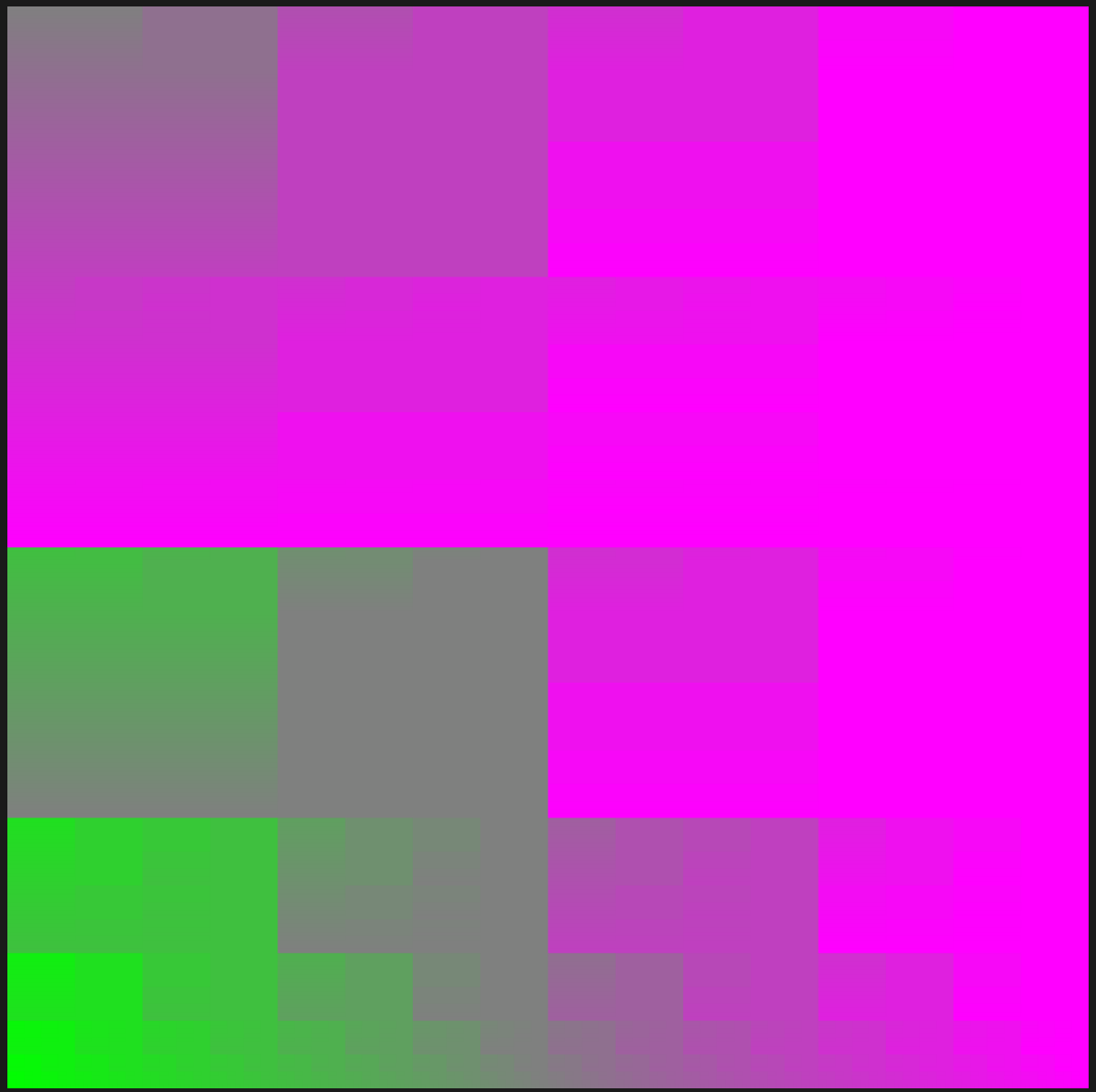
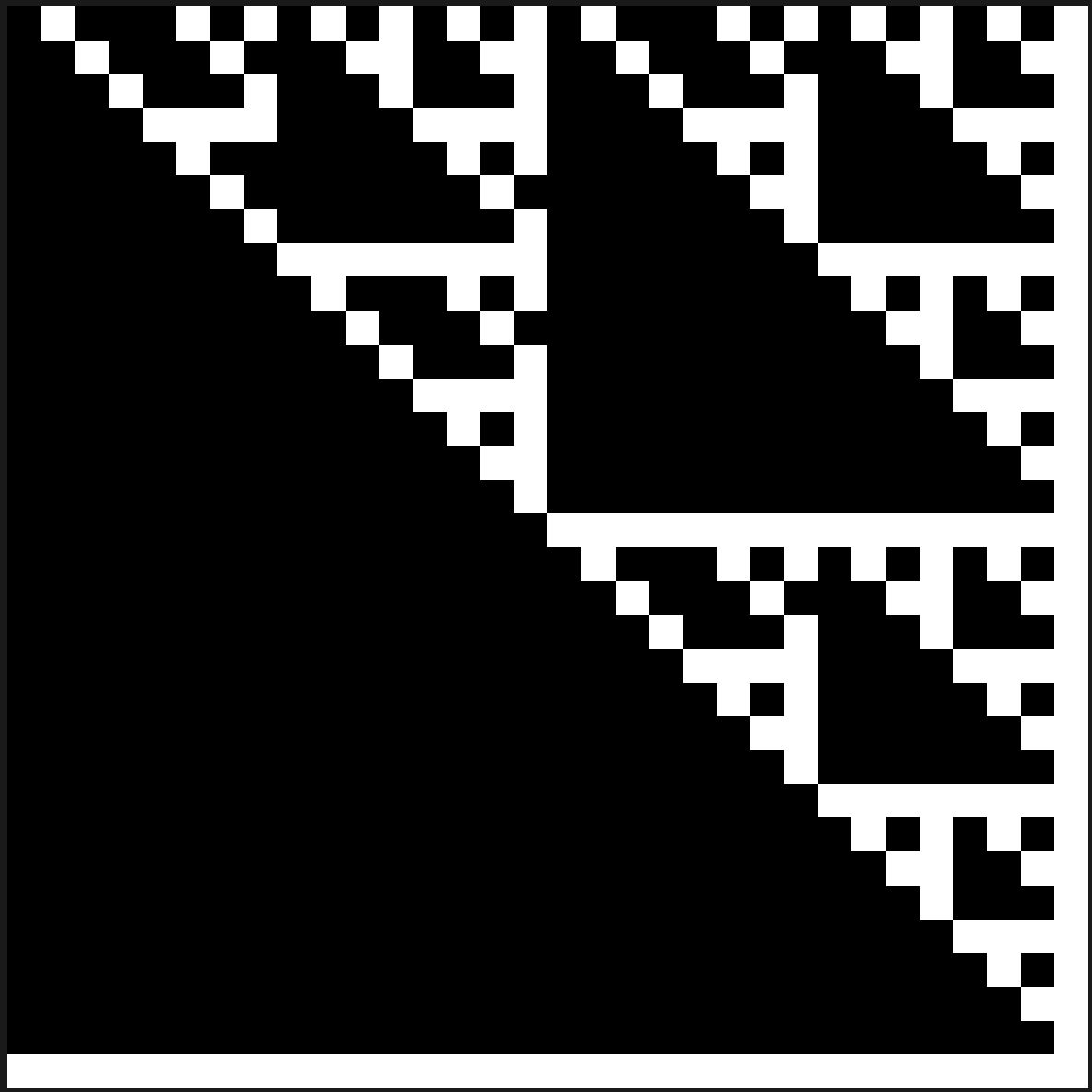


# Pictures: $FM_5^-$



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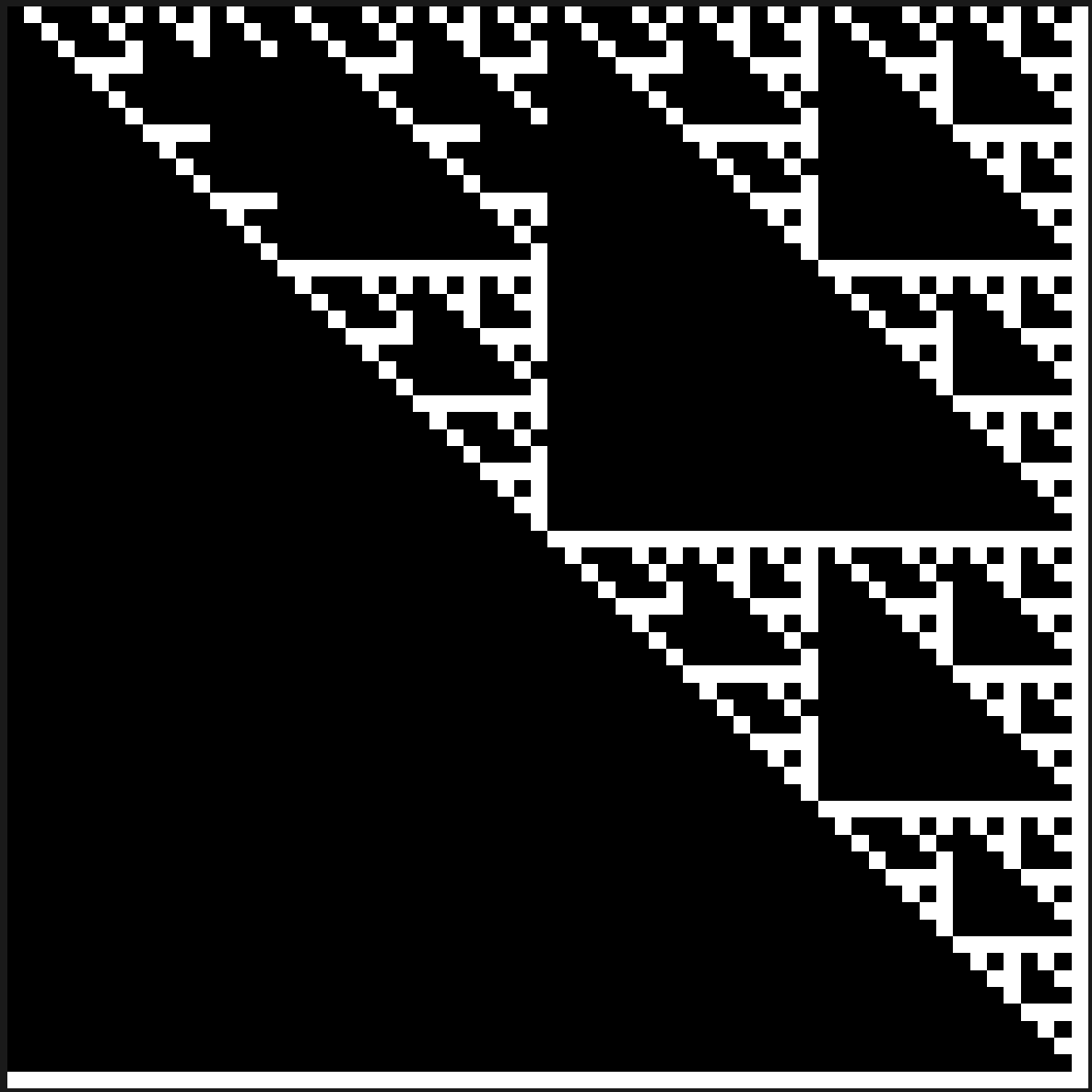

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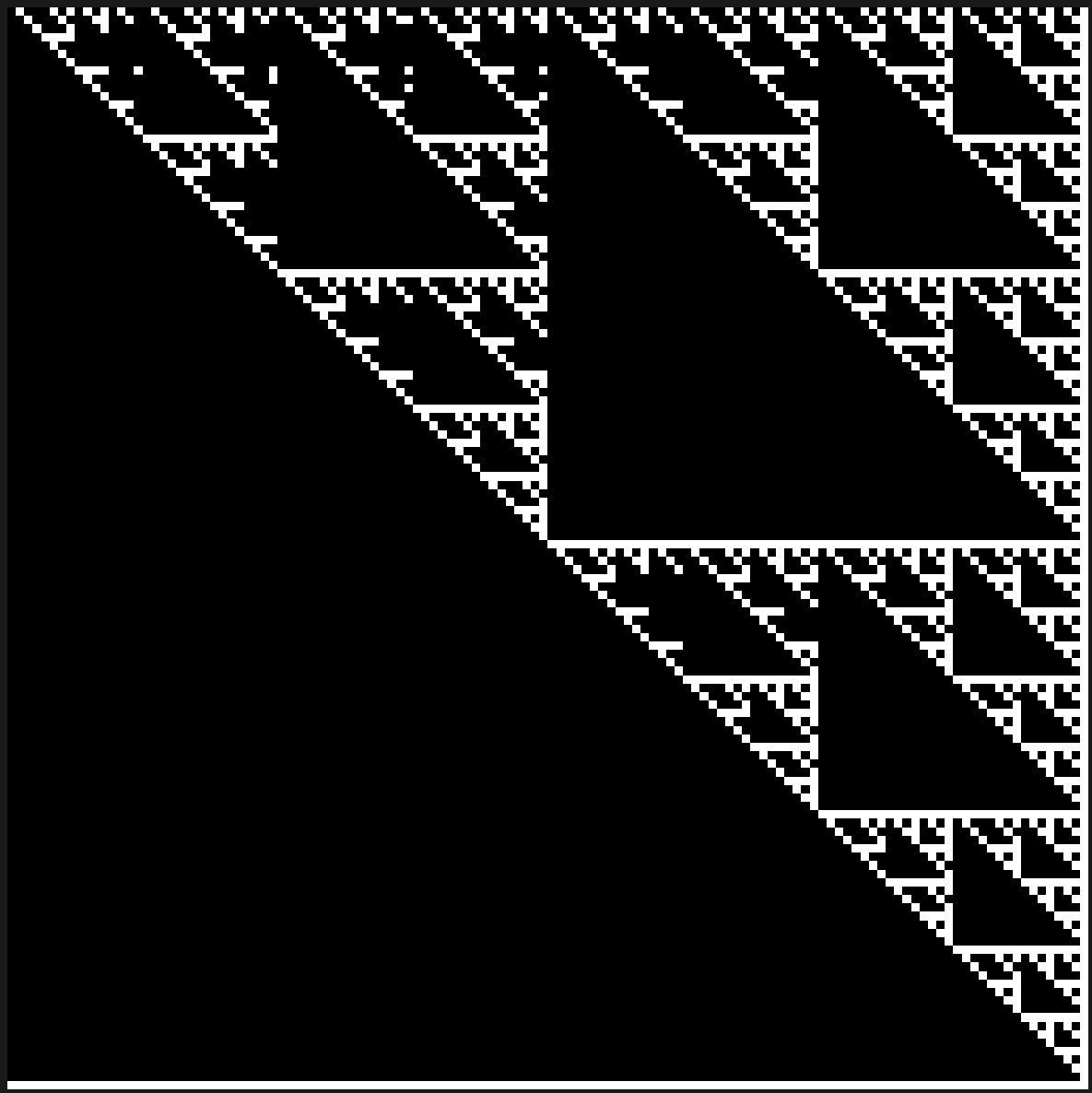

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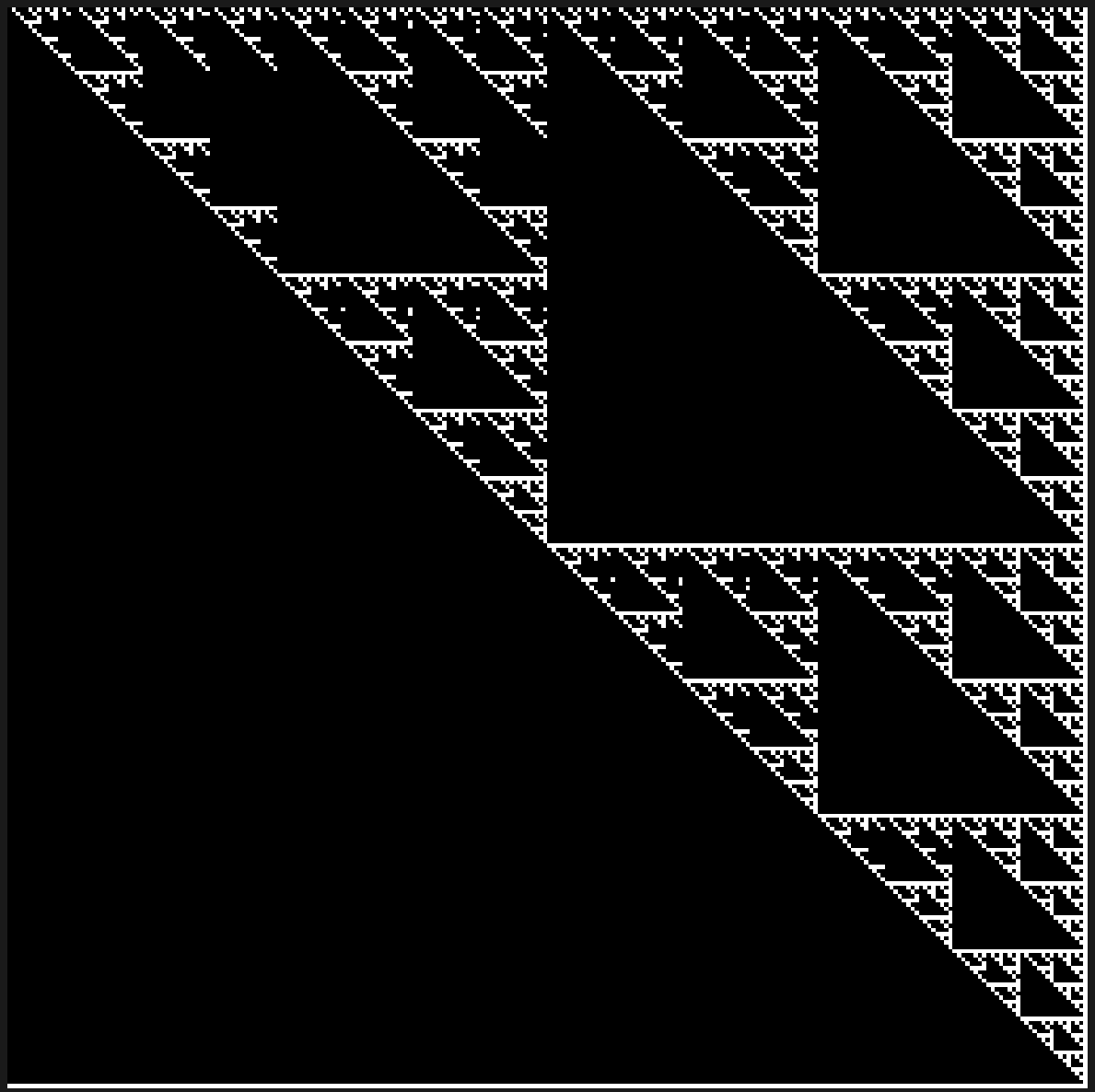
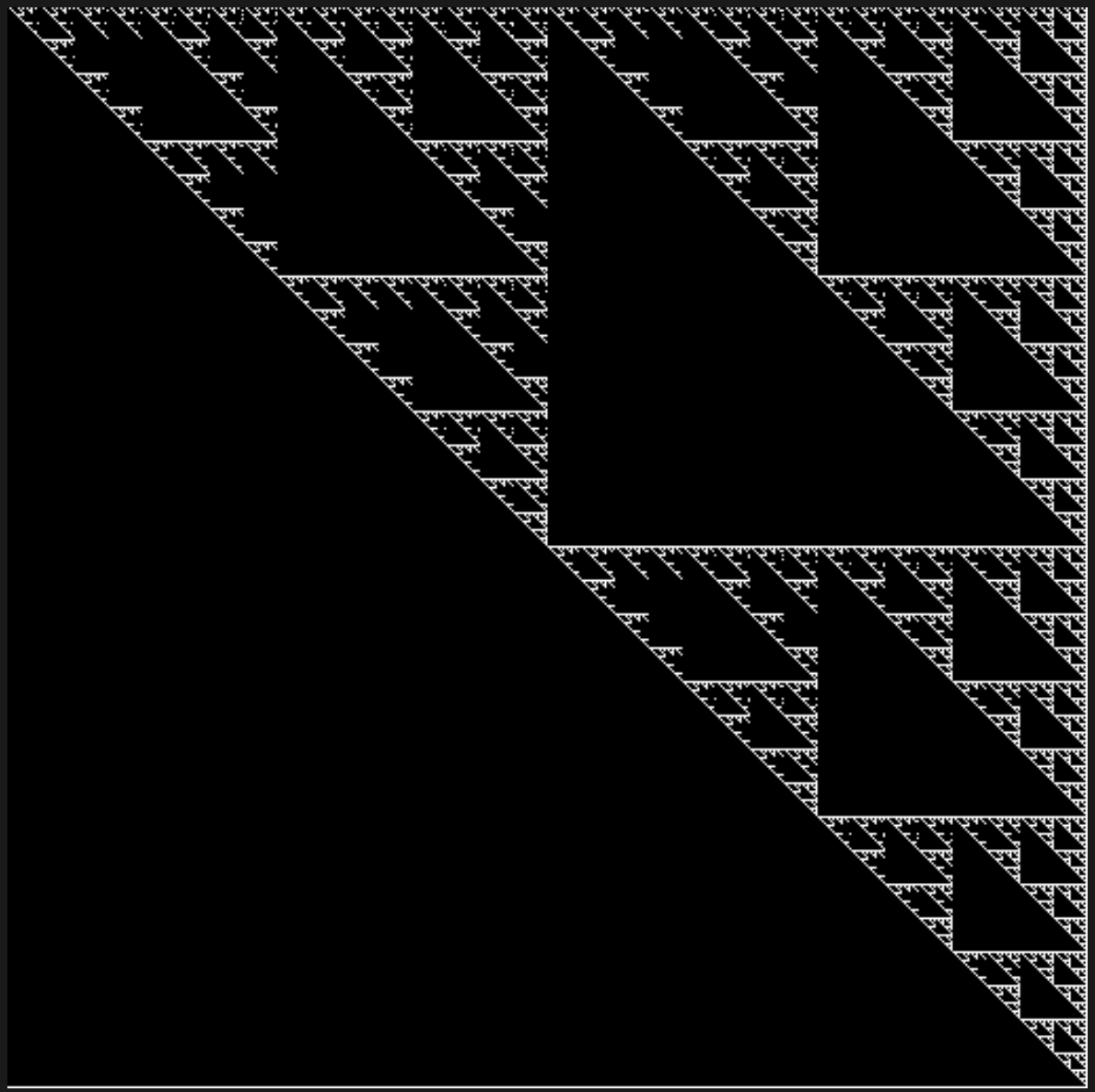
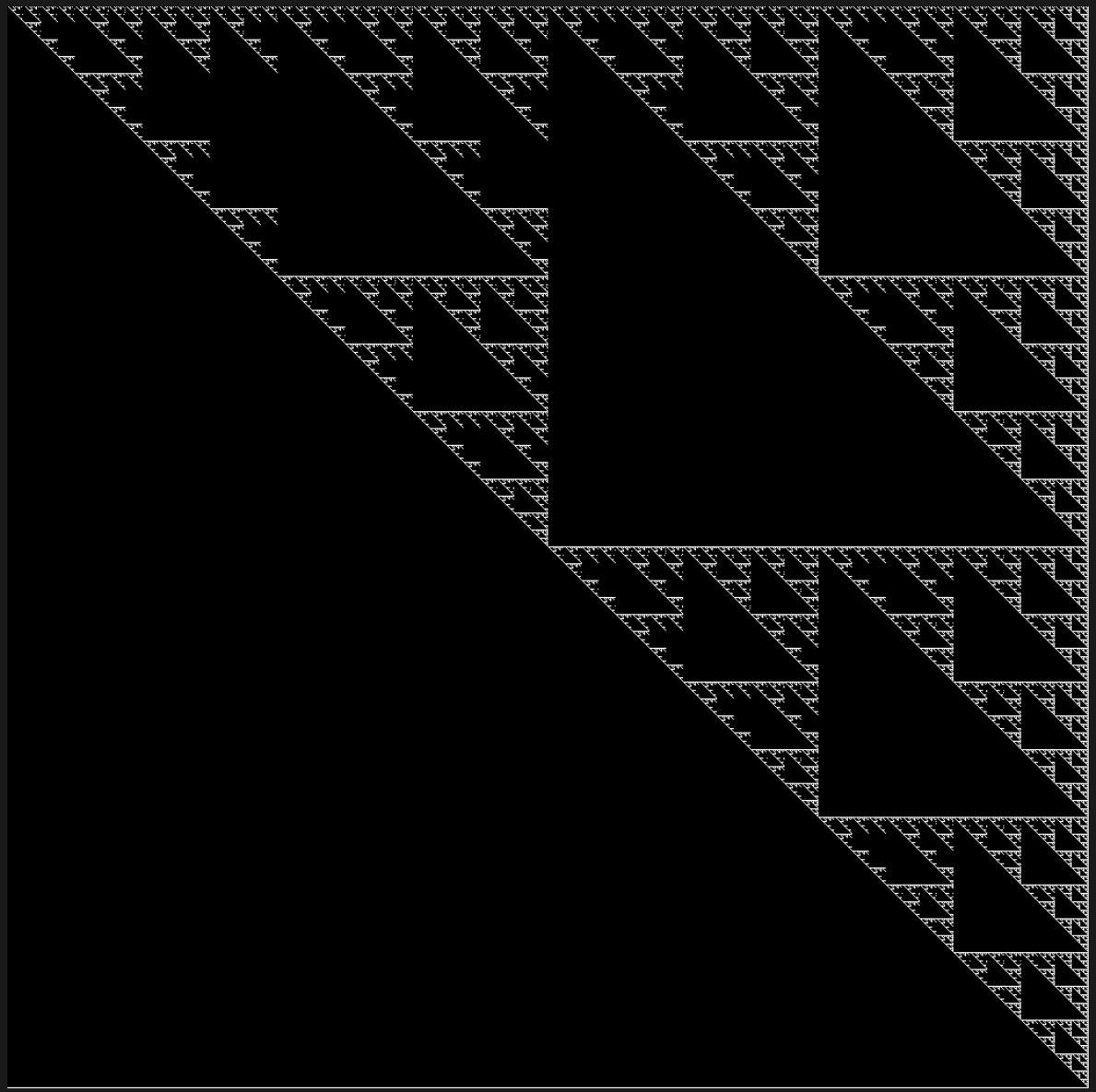


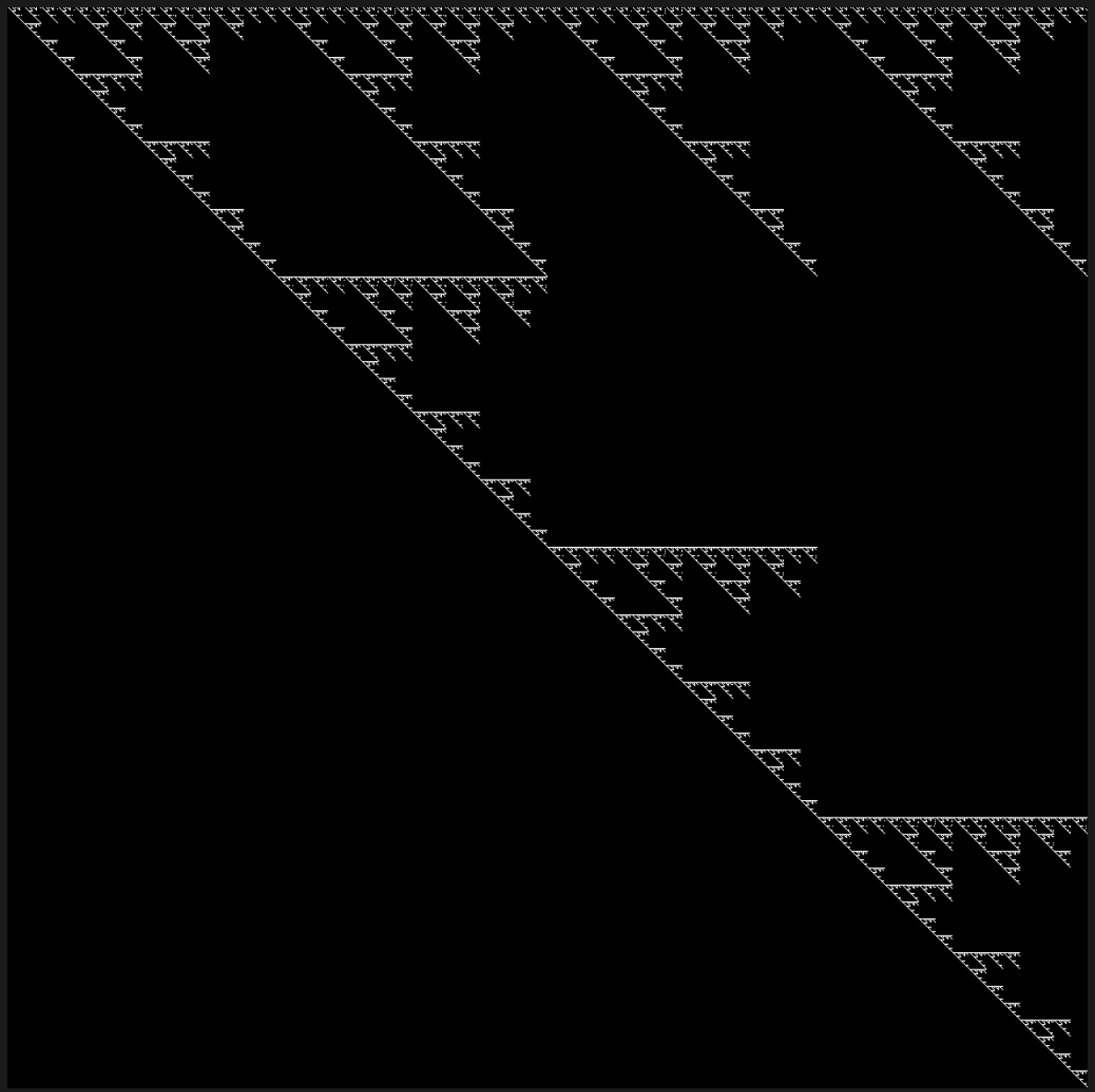
# Pictures: $FM_9^-$



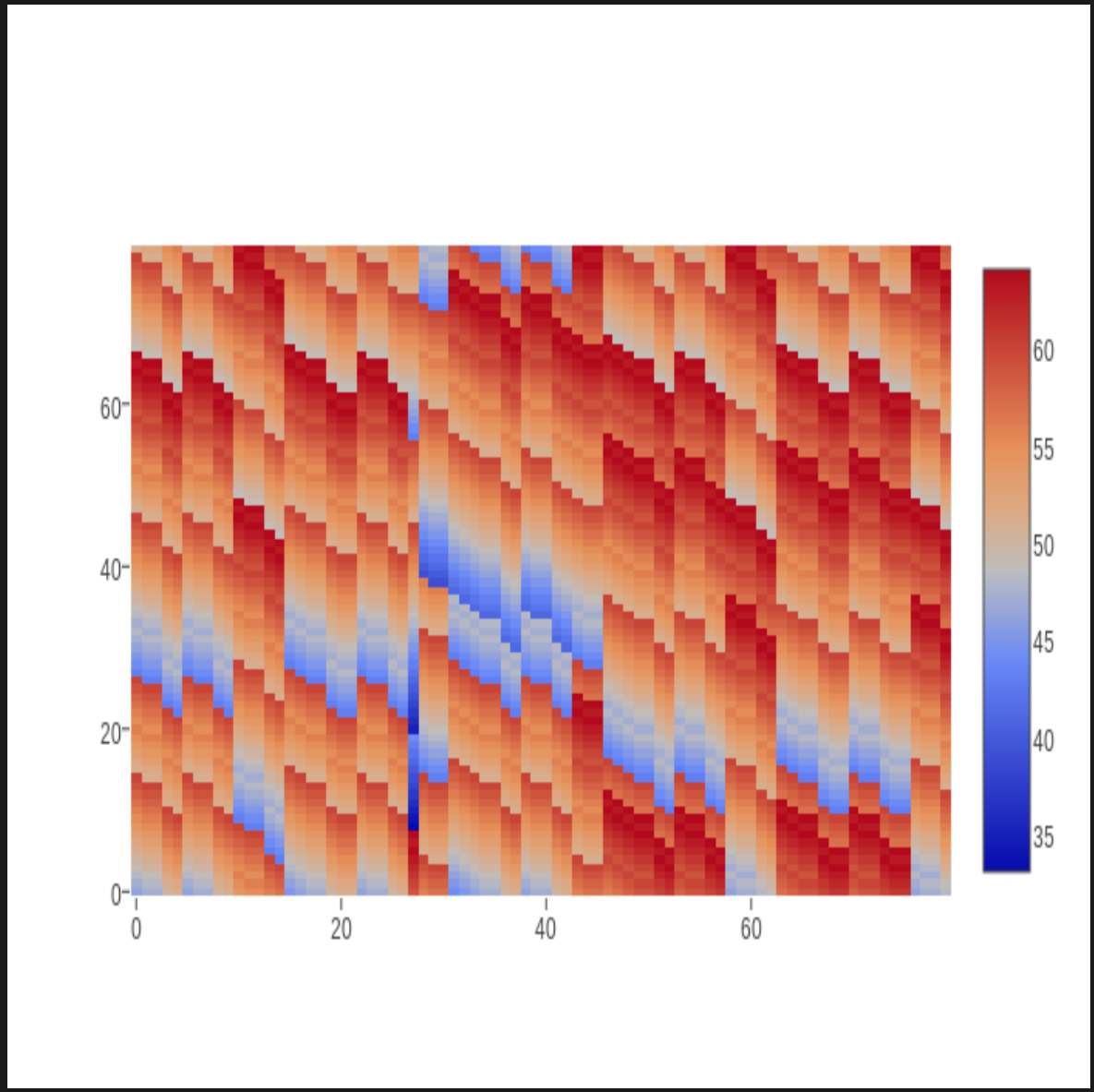
# Pictures: $FM_{10}^{-}$



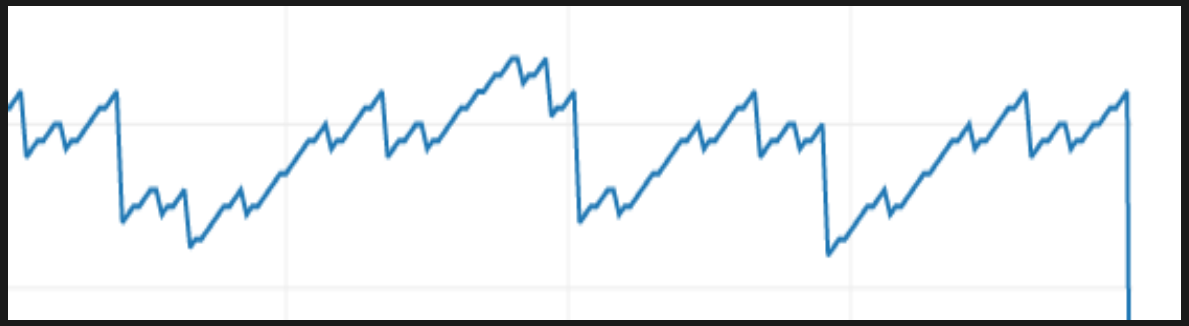
# Pictures:Multigenic Laver table snapshot



# Pictures:Endomorphic Laver table snapshot



# Pictures:Endomorphic Laver table snapshot



# The end

All work shall appear in my upcoming paper Generalizations of Laver tables (140 pages, abridged version available online).

JavaScript computer applications can be found at boolesrings.org/jvanname.