CLASSICAL GROUPS

NICK GILL

1. Abstract groups

Throughout this section G is a group.

1.1. **Simple groups.** The group G is called *simple* if the only normal subgroups of G are $\{1\}$ and G.

> **(E1)** *Prove that if G is a finite simple abelian group, then $G \cong C_p$, the cyclic subgroup of order p, where p is a prime.*

1.2. **Composition series and abelian series.** Let $H \leq G$. A *series from H to G* is a finite sequence $(G_i)_{0\leq i\leq k}$ of subgroups of G, such that

e: cs

$$H = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_k = G. \tag{1}$$

If H is unspecified, then you should assume that $H = \{1\}$. The sequence $(G_i)_{0\leq i\leq k}$ is called

- a *composition series* if, for $i = 1, \dots, k$, G_k/G_{k-1} is non-trivial and simple. The abstract group G_k/G_{k-1} is called a *composition factor* of G.
- an *abelian series* if for $i = 1, \dots, k$, G_k/G_{k-1} is abelian.
- a *normal series* if, for $i = 0, \dots, k$, $G_i \trianglelefteq G$.

Suppose that we have two series from H to G, the first given by (1), the second by:

second series

$$H = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_l = G. \tag{2}$$

Series (1) and (2) are called *equivalent* if $k = l$ and there exists a permutation $\pi \in S_k$ such that, for $i = 1, \dots, k$,

$$G_i/G_{i-1} \cong H_{i\pi}/H_{i\pi-1}.$$

The series (2) is said to be a *refinement* of series (1) if $k \leq l$ and there are non-negative integers $j_0 < j_1 < \cdots < j_k \leq l$ such that $G_i = H_{j_i}$ for $i = 0, \dots, k$.

Now the key result concerning series is due to Schreier [Ros94, 7.7]:

sition series

Lemma 1. *Any two series have equivalent refinements.*

> **(E2)** *Prove this.*

One important consequence of Lemma 1 is that if G is a group admitting a composition series, then the multiset of composition factors associated with any composition series of G is an invariant of the group G. In §2.4 we will briefly examine how, given M a finite multiset of simple groups, one might construct a group G for which M is the multiset of composition factors.

1.3. **Derived series.** For $g, h \in G$, define the *commutator* of g and h,

$$[g,h] := g^{-1}h^{-1}gh.$$

The *commutator subgroup*, or *derived subgroup* of G, written G' or $[G,G]$ or $G^{(1)}$, is the group

$$\langle [g,h] \mid g,h \in G\rangle.$$

Warning. G' is the group *generated* by all commutators of the group G, i.e. the smallest subgroup of G that contains all commutators. The set of all commutators in G is not necessarily a group.

: commutator

(E3*) *Prove that, for N a normal subgroup of G, the quotient G/N is abelian if and only if $G' \leq N$.*

(E4) *Find an example of a group G such that G' is not equal to the set of all commutators.*

We can generalize this construction as follows:

$$G^{(0)} := G;$$
$$G^{(n)} := [G^{(n-1)}, G^{(n-1)}] \text{ for } n \in \mathbb{N}.$$

We obtain a descending sequence of groups

$$\cdots \trianglelefteq G^{(2)} \trianglelefteq G^{(1)} \trianglelefteq G$$

which is called the *derived series* of G. If, for some k, $G^{(k)} = G^{(k+1)}$ then, clearly, $G^{(k)} = G^{(l)}$ for every $l \geq k$ and we say that the derived series *terminates* at $G^{(k)}$. Note that if the derived series does not terminate for any k then it is not strictly speaking a series. (Of course the derived series of a finite group always terminates.)

(E5) *Prove that (provided it terminates) the derived series is a normal series.*

We call G *perfect* if $G = [G,G]$. If G is finite, then the derived series terminates after k steps at a perfect group.

1.4. **Solvable groups.** We say that G is *soluble* or *solvable* if G has an abelian series.

(E6) *Prove that, if G is finite, then G is solvable if and only if all composition factors of G are cyclic of prime order. Give an example of a solvable group that does not have a composition series.*

(E7*) *Prove that a finite group G is solvable if and only if the derived series of G terminates at $\{1\}$.*