CLASSICAL GROUPS: DISCUSSION CLASS

The aim of this class is to discuss the (B, N)-structure of $\mathrm{GL}_n(k)$.[1] Let V be an n-dimensional vector space over a field k. Let $\{e_1, \dots, e_n\}$ be a basis for V and let $G = \mathrm{GL}_n(k)$.

(D1) The chain of subspaces

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \subset \cdots \subset \langle e_1, \dots, e_{n-1} \rangle$$

is called a *chamber*.

Let B be the stabilizer in G of this chamber. What is B?

> **Answer.** If we write our matrices with respect to the basis $\{e_1, \dots, e_n\}$, then B is the set of all upper triangular matrices. The group B and any conjugate of B is called a *Borel subgroup* of G.

(D2) Given a basis $\{e_1, \dots, e_n\}$, the corresponding *frame* is the set

$$\mathcal{F} = \{\langle e_1 \rangle, \langle e_2 \rangle, \dots, \langle e_n \rangle\}.$$

Let N be the stabilizer in G of the given frame. What is N?

> **Answer.** N is the set of all monomial matrices, that is all matrices with precisely on nonzero entry in each row and colum.

(D3) Show that $G = \langle N, B \rangle$.

> **Answer.** Let $g \in \mathrm{GL}_n(k)$ and let j be the last row such that $a_{j1} \neq 0$. For each $i < j$ premultiplying by a suitable transvection matrix $x_{ij}(\alpha) \in B$ is the elementary row operation $r_j \mapsto r_i + \alpha r_j$, so we can make a_{j1} the only nonzero entry in the first column.
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> Since A is invertible, there exists $j' \neq j$ such that $a_{j'2} \neq 0$. Take j' to be the last such row. Again by premultiplying by transvections from B we can make all entries in colum 2 except for those in rows j and j', equal to 0. Repeating this process we obtain a matrix h such that for each colum k there is a unique row whose first nonzero entry is in column k. Notice that $h = bg$ where $b \in B$.
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> Now there is a permutation matrix $n \in N$ such that nh is an upper triangular matrix, i.e. $nh = b' \in B$. We conclude that $nbg = b'$, i.e. $g = b'n^{-1}b^{-1}$. Since g was arbitrary we are done.

Remark: In fact our proof shows that $G = BNB$.

(D4) Let $H = B \cap N$. Show that H is a normal subgroup of N.

> **Answer.** Notice that H is the kernel of the action of N on $\mathcal{F}$, thus it must be a normal subgroup of N.

Remark: Note that H is the group of all diagonal matrices.

[1] This discussion class is taken from a course given by Michael Giudici. My thanks to him for letting me use it.

(D5) The group $W := N/H$ is called the *Weyl group* of G. What well-known group is N/H isomorphic to?

Answer. Let P be the subgroup of permutation matrices, i.e. the set of all matrices with one non-zero entry in each row and column, and all non-zero entries equal to 1.

Each $n \in N$ can be written as the product of a diagonal matrix in H with a permutation matrix from P. Thus $N = HP$ and, since $P \cap H = 1$, we obtain that

$$N/H = (HP)/H \cong P/(H \cap P) \cong P.$$

Now we claim that $P \cong S_n$, the symmetric group on n letters. The isomorphism is given by the map that takes each permutation $\sigma \in S_n$ to the matrix with a 1 in the (i,j)-entry if $i^\sigma = j$ and 0 in all other entries.

Remark: A *BN-pair* for a group G is a pair of subgroups B and N such that

(1) $G = \langle B, N \rangle$;
(2) $H = B \cap N \lhd N$;
(3) $W = N/H$ is generated by a set R of involutions such that, for $rH \in R$ and $n \in N$, then
 (3a) $rBnB \subset BnB \cup BrnB$;
 (3b) $rBr \neq B$.

We call $|I|$ the *rank* of the BN-pair

(D6) Let $R := \{(1,2), (2,3), (3,4), \dots, (n-1,n)\}$ a generating set of size $n-1$ for the group S_n. Prove that, with this generating set, (3a) and (3b) are satisfied for $GL_n(k)$, i.e. $GL_n(k)$ has a BN-pair.

Answer. (3b) is an easy matrix calculation. (3a) is a slightly more tricky matrix calculation that I leave for your edification.

Remark:

(1) It should be clear that, by taking the corresponding subgroups, we can see that $\mathrm{SL}_n(k)$, $\mathrm{PGL}_n(k)$ and $\mathrm{PSL}_n(k)$ also have BN-pairs.
(2) Tits has shown that given any group with a BN-pair, we can define a building on which G has a natural action. What is more, in this action, G is 'transitive on the pairs consisting of an apartment and a chamber contained in it' [Tit74, 3.2.6].
(3) Conversely Tits has shown that if a group G acts on a building so that it is 'transitive on the pairs consisting of an apartment and a chamber contained in it', then G has a BN-pair [Tit74, 3.11]. Thus the notion of a BN-pair and a building with this level of transitivity are closely linked.
(4) Finally Tits has shown that a finite building of 'irreducible type' and rank at least 3 is isomorphic to 'the building of a finite group of Lie type'. What is more such buildings admit transitive actions of the associated groups and we thereby have a full classification of those finite groups with a BN-pair of rank at least 3.
(5) Since the simple classical groups are 'groups of Lie type, they all have BN-pairs. Can you identify the groups B and N?

References

tits [Tit74] Jacques Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics, Vol. 386, Springer-Verlag, Berlin, 1974.