EXERCISE SHEET 0 WITH SOLUTIONS

(E3) Prove that, for N a normal subgroup of G, the quotient G/N is abelian if and only if $G' \leq N$.

Answer.

$$\begin{aligned} G/N \text{ is abelian} &\iff (gN)(hN) = (hN)(gN) \text{ for all } g, h \in G \\ &\iff g^{-1}h^{-1}gh \in N \text{ for all } g, h \in G \\ &\iff G' \leq N. \end{aligned}$$

(E7) Prove that a finite group G is solvable if and only if the derived series of G terminates at $\{1\}$.

Answer. If the derived series of G terminates at $\{1\}$, then the derived series is an abelian series for G and the group is solvable.

Conversely, suppose that

$$G = G_0 \geq G_1 \geq \cdots \geq G_k = \{1\}$$

is an abelian series for G. Observe that, since G_0/G_1 is abelian, G_1 contains $G^{(1)}$, the derived subgroup of G. Indeed, we can repeat the argument to observe that G_k contains $G^{(k)}$, the k-th term of the derived series. The result follows.