EXERCISE SHEET 2 WITH SOLUTIONS

Some solutions are sketches only. If you want more details, ask me!

(E35) Show that, for any prime power q, $\mathrm{PG}_2(q)$ is an abstract projective plane.

Answer. If $< u >$ and $< v >$ are distinct points in $\mathrm{PG}_2(q)$, then they are both incident with the plane $\langle u, v\rangle$, and with no other.

Any two planes passing through the origin in a 3-dimensional vector space must intersect in a subspace of dimension at least 1 (otherwise we would have four linearly independent vectors). If the planes are distinct, then the intersection has dimension exactly 1 as required.

$\mathrm{PG}_2(q)$ contains a quadrangle given by the points $\langle(0,0,1)\rangle, \langle(0,1,0\rangle, \langle(0,0,1)\rangle$ and $\langle(1,1,1)\rangle$, along with the lines with which they are incident.

(E39) Prove that the action of $\Gamma\mathrm{L}(V)$ on V is well-defined, and that $\Gamma\mathrm{L}(V)$ acts as a set of collineations of $\mathrm{PG}(V)$.

Answer. Well-defined: Suppose that $U = \langle u_1, \dots, u_k\rangle = \langle v_1, \dots, v_\ell\rangle$. Then $u_1 = \sum_{i=1}^{\ell} c_i v_i$ for some $c_i \in k$. Now, if $g \in \Gamma\mathrm{L}(V)$ with associated field automorphism σ, then

$$u_1^g = \sum_{i=1}^{\ell} c_i^\sigma v_i^g \in \langle v_1^g, \dots, v_\ell^g\rangle.$$

We conclude that

$$langle u_1^g, \dots, u_k^g\rangle \subseteq \langle v_1^g, \dots, v_\ell^g\rangle.$$

By symmetry, the reverse inclusion also holds, and our action is well-defined.

Acts as collineation: We must show that incidence is preserved, i.e. that if $U_1 < U_2 < V$, then $U_1^g < U_2^g < V$ for all $g \in \Gamma\mathrm{L}(V)$. This is obvious.

(E42) Prove that

$$|\mathrm{PGL}_n(\mathbb{R}) : \mathrm{PSL}_n(\mathbb{R})| = \begin{cases} 1, & \text{if } n \text{ is odd;} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

Answer. We note first that this question may have confused some at first because of our definitions:

$$\mathrm{PGL}_n(\mathbb{R}) = \mathrm{GL}_n(\mathbb{R})/K;$$
$$\mathrm{PSL}_n(\mathbb{R}) = \mathrm{SL}_n(\mathbb{R})/(K \cap \mathrm{SL}_n(\mathbb{R})).$$

Thus, as written $\mathrm{PSL}_n(\mathbb{R})$ is not a subgroup of $\mathrm{PGL}_n(\mathbb{R})$. However we can make use of the second isomorphism theorem of group theory to see an isomorphic copy of $\mathrm{PSL}_n(\mathbb{R})$ inside $\mathrm{PGL}_n(\mathbb{R})$:

$$\mathrm{PSL}_n(\mathbb{R}) = \mathrm{SL}_n(\mathbb{R})/(K \cap \mathrm{SL}_n(\mathbb{R}) \cong K\mathrm{SL}_n(\mathbb{R})/K.$$

In light of this remark the question reduces to calculating the index of $K\mathrm{SL}_n(\mathbb{R}) \in \mathrm{GL}_n(\mathbb{R})$. One must calculate the size of the set

$$\{\det(g) \mid g \in K\}.$$

But clearly this set is equal to

$$\{\{\alpha^n \mid \alpha \in k^*\}$$

and this set is equal to k^* whenever n is odd, and equal to the set of positive numbers if k is even. Since the latter is an index 2 subgroup in k^*, the result follows.

(E48) Prove that, for $n \geq 3$, $\mathrm{WAut}(\mathrm{PG}_n(q))$ contains $\mathrm{Aut}(\mathrm{PG}_n(q))$ as an index 2 subgroup. Can you say any more about the structure of $\mathrm{WAut}(\mathrm{PG}_n(q))$?

Answer. Observe that, for $1 \leq m, m' \leq n \geq 3$, we have $\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ m' \end{bmatrix}_q$ if and only if $m' \in \{m, n-m\}$. Thus spaces of dimension 1 are sent to spaces of dimension 1 or $n-1$. Suppose the former; now using Lemma 14 (2) we can see that spaces of dimension 2 must be sent to spaces of dimension 2, and so on. Thus weak automorphisms are either collineations or dualities. Now Proposition 16 implies that the set of dualities is a coset of the set of collineations inside the group of weak automorphisms, thus we conclude that $|\mathrm{WAut}(\mathrm{PG}_n(q)); \mathrm{Aut}(\mathrm{PG}_n(q))| = 2$ as required.

In fact one can prove that $\mathrm{WAut}(\mathrm{PG}_{n-1}(q) \cong \mathrm{P\Gamma L}_2(q) \rtimes \langle \iota \rangle$ where

$$\iota : \mathrm{PSL}_n(q) \to \mathrm{PSL}_n(q), x \mapsto x^{-T}.$$

See (E59) by way of comparison.

(E50) Prove that the action of $\mathrm{PGL}(V)$ on Σ_V is regular.

Answer. We know that $\mathrm{PGL}(V)$ acts transitively on Σ_V, thus it is enough to show that the stabilizer in $\mathrm{GL}(V)$ of a point of Σ_V is the group K. Take the special tuple $(e_1, \dots, e_n, \sum_{i=1}^n +e_i)$ where $\{e_1, \dots, e_n\}$ is a fixed basis for V. The stabilizer of the first n-entries of the tuple is clearly

$$\{\mathrm{diag}(\lambda_1, \dots, \lambda_n) \mid \lambda_1, \dots, \lambda_n \in k\}.$$

Now the stabilizer in this group of $\sum_{i=1}^n +e_i$ is clearly the group K as required.

(E51) Prove that $\mathrm{PSL}_n(k)$ is 2-transitive on the points of $\mathrm{PG}_{n-1}(k)$. Prove, furthermore, that $\mathrm{PSL}_n(k)$ is 3-transitive if and only if $n = 2$ and every element of k is a square.

Answer. The first part was done in lectures. Now suppose that $\mathrm{PSL}_n(k)$ is 3-transitive on points of $\mathrm{PG}_{n-1}(k)$. If $n \geq 3$, then this would imply that $\mathrm{PSL}_n(k)$ mapped a triple of vectors generating a 3-dimensional space to a triple of vectors generating a 2-dimensional vector space. This is a contradiction, hence we conclude that $n = 2$. In this case, let e_1, e_2 be a basis. 3-transitivity implies that the stabilizer of the pair $(\langle e_1\rangle, \langle e_2\rangle)$ is transitive on the remaining 1-subspaces. This stabilizer is equal to

$$S = \{\mathrm{diag}(\lambda, \lambda^{-1} \mid \lambda \in k\}.$$

Clearly the orbit of $\langle(1,1)\rangle$ is equal to the set of 1-spaces $\langle(c,d)\rangle$ where c/d is a non-zero square. Thus we conclude that all non-zero elements of k are square and we are done.

Conversely if $n = 2$ and every element of k is a square, then it is clear that S is transitive on all 1-subspaces apart from $\langle e_1\rangle$ and $\langle e_2\rangle$. The result follows.

(E52) Let $G = \mathrm{GL}_n(k)$ and $\omega \in \Omega$, the set of points of $\mathrm{PG}(V)$. Then

$$G_\omega \cong Q.GL_{n-1}(k)$$

where Q is an abelian group isomorphic to the additive group $(k^{n-1}, +)$. Prove that the extension is split.

Answer. Simply observe that $G_\omega = QR$, a product of two groups, with

$$G_{\langle e_n\rangle} = \left\{ g := \left(\begin{array}{ccc|c} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & a \end{array}\right) \;\middle|\; \begin{array}{l} a \in k^*, \\ A \in \mathrm{GL}_{n-1}(k), \\ a = \frac{1}{\det(A)} \end{array} \right\}.$$

Since Q is normal in G_ω and $Q \cap R = \{1\}$, every element of G_ω can be written in a unique way as a product of an element from Q and an element from R. The result follows (cf. the answer to (E33) which uses the same method).

(E53) Prove that if $n \geq 3$, then $\mathrm{SL}_n(k)$ contains a unique conjugacy class of transvections. Prove that if $n = 2$, then $\mathrm{SL}_n(k)$ contains one or two conjugacy classes of transvections. Can you characterise when $\mathrm{SL}_n(k)$ contains two conjugacy classes, and describe how the subgroup Q intersects each class? (In particular you should show that each class has non-empty intersection with Q.)

Answer. We proved in lectures that all transvections lie in $\mathrm{SL}_n(k)$ and that they are all conjugate in $\mathrm{GL}_n(k)$. Thus, given a transvection t, there is a matrix $g \in \mathrm{GL}_n(k)$ such that

$$gtg^{-1} = t_0 := \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

Now let $n \geq 3$ and observe that, for $a, b \in k^*$, the matrix

$$h := \mathrm{diag}(a, b, 1, \ldots, 1, a)$$

centralizes t_0. Thus $hgtg^{-1}h^{-1} = t_0$ and $\det(hg) = a^2 \cdot b \cdot \det(g)$. Now choose $b = 1/(a^2 \cdot \det(g))$ and we have a matrix in $\mathrm{SL}_n(q)$ conjugating t to t_0 as required.

If $n = 2$, then the matrix

$$h := \mathrm{diag}(a, a)$$

centralizes t_0. Thus $hgtg^{-1}h^{-1} = t_0$ and $\det(hg) = a^2 \cdot \det(g)$. If every element of k is a square (e.g. if k is finite and $\mathrm{char}(k) = 2$), then there is a choice of h for which $\det(hg) = 1$, and there is one conjugacy class of transvections. On the other hand if there are elements of k which are non-squares (e.g. if k is finite and $\mathrm{char}(k) \neq 2$), then one cannot conjugate the following transvection to t:

$$t_1 := \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}.$$

(Here c is a non-square in k.) On the other hand it is clear that every transvection can be conjugated to either t_0 or t_1, so the result follows.

(E54) Let t be a transvection in $\mathrm{SL}_n(k)$ with $|k| \leq 3$. Prove that t is a commutator except when $n = 2$.

Answer. See page 21 of Cameron's notes on "Classical Groups."

(E55) Show that the set of upper-triangular matrices with 1's on the diagonal is a Sylow p-subgroup of $\mathrm{GL}_n(q)$.

Answer. Simply compare the order formula for $\mathrm{GL}_n(q)$ given in Proposition 26 of lectures, with the order of the set of upper-triangular matrices ($q^{\frac{1}{2}n(n-1)}$). The result follows immediately.

(E56) Prove that that the incidence structure defined in Proposition 27 is isomorphic to the Fano plane, and that the natural conjugation action of G on the conjugates of U and V respectively, induces an action on $\mathcal{I}$.

Answer. Here is a Sylow 2-subgroup of $\mathrm{SL}_2(7)$:

$$S := \langle \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle.$$

The first matrix has order 8, the second has order 4. Their projective images in $\mathrm{PSL}_2(7)$ have orders 4 and 2 respectively, and they generate a dihedral group of order 8. Since 8 divide $\mathrm{PSL}_2(7)$ but 16 does not, this must be a Sylow 2-subgroup of $\mathrm{PSL}_2(7)$.

It is easy to check that a dihedral group of order 8 contains two Klein 4-groups that are not conjugate to each other. Next observe that the Klein 4-group U

which is the projective image of

$$\langle \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle$$

is normalized by the projective image of

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Thus, since U is normal in S, we get $|N_G(U)| \geq 24$. Thus $|N_G(U)| = 24$ or 168, but the latter contradicts the simplicity of G, so we obtain that $|N_G(U)| = 24$ and there are seven conjugates of U. A similar calculation implies that there are seven conjugates of the other Klein 4-group, V, in S, and that U and V are not conjugate. Now since both $N_G(U)$ and $N_G(V)$ contain 3 Sylow 2-subgroups we conclude that each conjguate of U is incident to three conjugates of V, and vice-versa. The result follows easily.

(E57) Prove that $\mathrm{PSL}_3(4) \not\cong \mathrm{SL}_4(2) \cong A_8$, despite the fact that these groups have the same orders.

Answer. Recall that an **involution** in a group is an element of order 2. It is easy to see that A_8 has two conjugacy classes of involutions - one whose elements fix four points, one whose elements fix zero points. We will prove that $\mathrm{PSL}_3(4)$ has a single conjugacy class of involutions and the result will follow.

Since every involution lies in a Sylow 2-subgroup, all of which are conjugate, we need only show that all involutions in any given Sylow 2-subgroup are conjugate to each other. By (a variant of) (E55) we choose the Sylow 2-subgroup equal to upper-triangular matrices and observe that involutions have the following form:

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

where $a, b \in k^*$. Now involutions of the first kind are all conjugate to each other, because we can choose a conjugating matrix in the group

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \middle| A \in \mathrm{GL}_2(q), a = 1/\det(A) \right\}$$

which does the trick. A similar argument shows that all involutions of the second kind are conjugate to each other. Since the two kinds overlap we are done.

(E59) Prove that

$$\mathrm{Aut}(\mathrm{PSL}_n(q)) \geq \begin{cases} \mathrm{P\Gamma L}_2(q), & \text{if } n = 2; \\ \mathrm{P\Gamma L}_2(q) \rtimes \langle \iota \rangle, & \text{if } n \neq 3. \end{cases}$$

Hint: you need to study the natural action of, say, $\mathrm{P\Gamma L}_n(q)$ on its normal subgroup $\mathrm{PSL}_n(q)$.

Answer. Let K be the unique normal subgroup of $G = \mathrm{P\Gamma L}_n(q)$ (resp. $G = \mathrm{P\Gamma L}_2(q) \rtimes \langle \iota \rangle$) that is isomorphic to $\mathrm{PSL}_n(q)$. To prove the result it is sufficient to show that $C_G(K) = \{1\}$.

Warning: In what follows I consider centralizers of matrices (i.e. elements of $\mathrm{SL}_n(q)$) rather than their images in $\mathrm{PSL}_n(q)$. One needs to be a little careful about how you do this, as the centralizers of their images can be larger. I will skim these details though - if you want me to explain more, please ask.

Consider the set of matrices in $\mathrm{SL}_n(q)$ whose entries are in the prime field $\mathbb{F}_p$. This is a subgroup isomorphic to $\mathrm{SL}_n(p)$. The centralizer in $\mathrm{PGL}_n(q)$ of the projective image of this subgroup in $\mathrm{P\Gamma L}_n(q)$ can easily shown to equal $\mathrm{Aut}(\mathbb{F}_q)$. But now any (projective image of a) matrix with elements outside all proper subfields of $\mathbb{F}_q$ will have trivial centralizer in $\mathrm{Aut}(\mathbb{F}_q)$. Thus we conclude that $C_{\mathrm{P\Gamma L}_n(q)}(\mathrm{PSL}_n(q))$ is trivial.

This proves the result for $n = 2$. For $n \geq 3$, it implies that $C_G(K)$ has size at most 2 (since if it was larger it would intersect $\mathrm{P\Gamma L}_n(q)$ non-trivially). Now suppose that $|k|$ is odd and let

$$g = \mathrm{diag}(1, 1, \dots, 1, a, a^{-1})$$

Then

$$g^\iota = \mathrm{diag}(1, 1, \dots, 1, a^{-1}, a)$$

Since $|k|$ is odd, one can choose a such that $a^{-1} \neq a^{p^x}$ for any $x \in \mathbb{N}$ and thus g and g^ι are conjugate in $\mathrm{P\Gamma L}_n(q)$ only by matrices of the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & 0 & a \\ 0 & b & 0 \end{pmatrix}$$

where $A \in \mathrm{GL}_{n-2}(q)$ and $a, b \in k^*$. Choosing variations on g where the a and a^{-1} are in different positions on the diagonal, one quickly concludes that no element of $\mathrm{P\Gamma L}_n(q)$ simultaneously conjugates all such g's to g^ι. This implies that any element in G centralizing g must lie in $\mathrm{P\Gamma L}_n(q)$, and we conclude that $C_G(g)$.

We leave the case when $|k|$ is even as an(other) exercise.