EXERCISE SHEET 2

Answers should be handed in by Monday 2nd November 2009.

1. Let $\mathbb{Q}(\alpha) : \mathbb{Q}$ be a simple field extension with the property that the minimal polynomial of α is $t^3 + t^2 - 2$. Calculate the minimal polynomials of $\alpha - 1$ and $\alpha^2 + 1$ over $\mathbb{Q}$, and express their multiplicative inverses in $\mathbb{Q}(\alpha)$ in the form $c_0 + c_1\alpha + c_2\alpha^2$ for suitable $c_0, c_1, c_2 \in \mathbb{Q}$.

2. Suppose that $L : K$ is a field extension, and that $\alpha \in L$ is algebraic over K, and $\beta \in L$ is transcendental over K. If, in addition, one has $\alpha \notin K$, show that $K(\alpha, \beta) : K$ is not a simple extension.

3.

(a) Show that if p is a prime number, then for every positive integer n the polynmomial $x^n - p$ is irreducible over $\mathbb{Q}[x]$.
(b) By making the substitution $y = x - 1$, or otherwise, show that when p is a prime number, the polynomial $x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible over $\mathbb{Q}$.

4.

(a) Let K be a field of characteristic not equal to 2, and let m be a quadratic polynomial over K. Show that m has a zero in a simple extension $K(\alpha)$ of K where $\alpha^2 = k \in K$. thus, allowing "square roots" $\sqrt{k}$ allows us to solve all quadratic equations over K.
(b) Show that for fields of characteristic 2 there exist quadratic equations which cannot be solved by adjoining square roots of elements of the field. (Hint: consider $\mathbb{Z}/2\mathbb{Z}$.
(c) Show that we can solve all quadratic equations over a field of characteristic 2 if we allow ourselves not only to adjoin square roots of elements but generalized square roots $\sqrt[*]{k}$ defined to be solutions of the equation $t^2 + t = k$.

5. Let $m \in \mathbb{Z}^+$ and $U_m = \{z \in \mathbb{C} \mid z^m = 1\}$. Then U_m is a cyclic subgroup of $\mathbb{C}^*$ of order m, generated by $\omega = e^{2\pi i/m}$. Any generator of U_m is a called a *primitive m-th root of* 1; these are the ω^k where$(k, m) = 1$, so there are $\phi(m)$ primitive m-th roots of 1, where ϕ is *Euler's ϕ-function.*

The field $\mathbb{Q}_m = \mathbb{Q}(\omega)$ is called the *m-th cyclotomic field*; m_ω, the minimal polynomial of ω over $\mathbb{Q}$, is called the *m-th cyclotomic polynomial.* Prove the following:

(a) By considering derivatives prove that m_ω has no multiple zeros.
(b) Show that a zero of m_ω is a primitive m-th root of 1.
(c) If θ is a primitive m-th root of 1 and p is prime with $(m, p) = 1$, then θ^p has the same minimum polynomial as θ.
(d) Show that every primitive m-th root of 1 is a zero of m_ω, and conclude that $\deg(m_\omega) = \phi(m)$.