EXERCISE SHEET 3

Answers should be handed in by Monday 16th November 2009.

1. For each of the following polynomials, construct a splitting field L over $\mathbb{Q}$ and compute $[L:\mathbb{Q}]$.
(i) t^3-1; (ii)t^4+5t^2+6; (iii) t^6-8.

2. Let K be a field, and suppose that $f \in K[t]$ is **not** irreducible, and that amongst its irreducible factors are two of different degrees. Let $\Sigma : K$ be a splitting field extension for f. Show that there are roots α and β of f with the property that there exists no automorphism $\sigma : \Sigma \to \Sigma$ that fixes K and satisfies $\sigma(\alpha) = \beta$. What happens in the situation where the irreducible factors of f have the same degree?

3. Take $a \in \mathbb{F}_p$ and suppose that $f(t) = t^p - t - a$ is irreducible in $\mathbb{F}_p[t]$.

(a) Show that if α is a zero of $f(t)$, then $\alpha + i$ is also a zero for each $i \in \mathbb{F}_p$;
(b) Deduce that $\Gamma(\mathbb{F}_p(\alpha) : \mathbb{F}_p)$ is cyclic of order p, and find a generator.

4. Suppose that $L : K$ is a field extension, and that for $i = 1, 2$, the field M_i is an intermediate field with $K \subseteq M_i \subseteq L$. Show that if $M_i : K$ is normal for $i = 1$ and 2, then (i) $K(M_1, M_2) : K$ is normal, and (ii) $M_1 \cap M_2 : K$ is normal.

5. Let K be a field of characteristic 0, and $L : K$ an extension of degree 2. Prove that $L : K$ is a normal extension. Now prove the same result for K of characteristic 2.

6. Which of the following extensions are normal? (Give reasons).

(a) $\mathbb{Q}(t) : \mathbb{Q}$;
(b) $\mathbb{Q}(\sqrt{-5}) : \mathbb{Q}$;
(c) $\mathbb{Q}(\alpha) : \mathbb{Q}$, where α is the real seventh root of 5;
(d) $\mathbb{Q}(\sqrt{5}, \alpha) : \mathbb{Q}(\alpha)$, where α is the real seventh root of 5;
(e) $\mathbb{R}(\sqrt{-7}) : \mathbb{R}$.