10. Fields and Vector Spaces

(E10.1) Check that $\mathbb{H}$ is a division ring.

(E10.2) Show that Vandermonde's Theorem does not hold in $\mathbb{H}$.

(E10.3) Let k be finite of order n. How many primitive elements does k contain?

(E10.4*) Show that $X^2+1 \in \mathbb{F}_3[X]$ is irreducible, and compute the addition and multiplication tables for $\mathbb{F}_9 := \mathbb{F}_3[x]/\langle X^2+1\rangle$.

(E10.5*) Show that $X^3+X+1 \in \mathbb{F}_2[X]$ is irreducible, and compute the addition and multiplication tables for $\mathbb{F}_8 = \mathbb{F}_2[x]/\langle X^3+X+1\rangle$.

(E10.6) Fix a basis B of V. Any semilinear transformation on V is a composition of a linear transformation and a field automorphism of V with respect to B.

(E10.7) Use the previous exercise to prove that $\Gamma\mathrm{L}_n(k) \cong \mathrm{GL}_n(k) \rtimes_\phi \mathrm{Aut}(k)$. Note that this implicitly (or explicitly) requires an appropriate homomorphism $\phi : \mathrm{Aut}(k) \to \mathrm{Aut}(\mathrm{GL}_n(k))$... Can you speculate about the structure of $\mathrm{Aut}(\mathrm{GL}_n(k))$ when k is finite?