9. Series

9.1. **Composition series and abelian series.** Let $H \leq G$. A *series from H to G* is a finite sequence $(G_i)_{0\leq i\leq k}$ of subgroups of G, such that

$$H = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_k = G. \tag{8}$$

We call a sequence a *series for G* if it is a series from $\{1\}$ to G.

Consider a series $(G_i)_{0\leq i\leq k}$ for a group G. We say that the series has *length* k, and we call it

- a *composition series* if, for $i = 1, \ldots, k$, G_k/G_{k-1} is non-trivial and simple. The abstract group G_k/G_{k-1} is called a *composition factor* of G.
- an *abelian series* if for $i = 1, \ldots, k$, G_k/G_{k-1} is abelian.
- a *normal series* if, for $i = 1, \ldots, k$, $G_i \trianglelefteq G$.
- a *central series* if it is a normal series and, for $1, \ldots, k$, G_i/G_{i-1} is central in G/G_{i-1}.

Suppose that we have two series from H to G, the first given by (8), the second by:

$$H = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_l = G. \tag{9}$$

Series (8) and (9) are called *equivalent* if $k = l$ and there exists a permutation $\pi \in S_k$ such that, for $i = 1, \ldots, k$,

$$G_i/G_{i-1} \cong H_{i\pi}/H_{i\pi-1}.$$

The series (9) is said to be a *refinement* of series (8) if $k \leq l$ and there are non-negative integers $j_0 < j_1 < \cdots < j_k \leq l$ such that $G_i = H_{j_i}$ for $i = 0, \ldots, k$.

Now the key result concerning series is due to Schreier [Ros94, 7.7]:

Lemma 9.1. *Let G be a finite group. Any two series have equivalent refinements.*

> **(E9.1)** *Prove this. (This is hard.)*

A corollary of Lemma 9.1 is the Jordan-Hölder theorem:

Corollary 9.2. *If G is finite, then any two composition series are equivalent.*

> **(E9.2)** *Prove this.*

Corollary 9.2 implies, in particular, that the multiset of composition factors associated with any composition series of a finite group G is an invariant of G.

9.2. **Derived series.** For $g, h \in G$, define the *commutator* of g and h,

$$[g, h] := g^{-1}h^{-1}gh.$$

The *commutator subgroup*, or *derived subgroup* of G, written G' or $[G, G]$ or $G^{(1)}$, is the group

$$\langle [g, h] \mid g, h \in G\rangle.$$

Warning. G' is the group *generated* by all commutators of the group G, i.e. the smallest subgroup of G that contains all commutators. The set of all commutators in G is not necessarily a group.

> **(E9.3)** *Prove that, for N a normal subgroup of G, the quotient G/N is abelian if and only if $G' \leq N$.*
>
> **(E9.4)** *Find an example of a group G such that G' is not equal to the set of all commutators. (This is tricky; if you know about free groups, then I'd start there...)*

We can generalize this construction as follows.

$$G^{(0)} := G;$$
$$G^{(n)} := [G^{(n-1)}, G^{(n-1)}] \text{ for } n \in \mathbb{N}.$$

We obtain a descending sequence of groups

$$\cdots \trianglelefteq G^{(2)} \trianglelefteq G^{(1)} \trianglelefteq G$$

which is called the *derived series* of G. If, for some k, $G^{(k)} = G^{(k+1)}$ then, clearly, $G^{(k)} = G^{(l)}$ for every $l \geq k$ and we say that the derived series *terminates* at $G^{(k)}$. Note that if the derived series does not terminate for any k then it is not strictly speaking a series. (Of course the derived series of a finite group always terminates.)

(E9.5) *Prove that (provided it terminates) the derived series is a normal series.*

We call G *perfect* if $G = [G, G]$. If G is finite, then the derived series terminates after k steps at a perfect group.

9.3. **Solvable groups.** We say that G is *soluble* or *solvable* if G has an abelian series.

(E9.6) *Prove that, if G is finite, then G is solvable if and only if all composition factors of G are cyclic of prime order. Give an example of a solvable group that does not have a composition series.*

(E9.7) *Prove that a finite group G is solvable if and only if the derived series of G terminates at $\{1\}$.*

9.4. **Nilpotent groups.** We say that G is *nilpotent* if G has a central series. The *nilpotency class* of G is the minimum integer n for which G has a central series

$$\{1\} = G_0 < G_1 < \cdots < G_n.$$

(E9.8) *What is another name for a nilpotent group of class 1?*

(E9.9) *Prove that a p-group is nilpotent.*

Nilpotent groups have two alternative definitions, as the next two exercises will make clear. For two subgroups $H, K \leq G$ define

$$[H, K] = \langle [h, k] \mid h \in H, k \in K \rangle.$$

Note that this is consistent with our definition of $[G, G]$. Now define a sequence of groups as follows:

$$G^{[0]} := G;$$
$$G^{[n]} := [G^{[n-1]}, G] \text{ for } n \in \mathbb{N}.$$

We obtain a descending sequence of groups

$$\cdots \trianglelefteq G^{[2]} \trianglelefteq G^{[1]} \trianglelefteq G$$

which is called the *lower central series* of G. If, for some k, $G^{[k]} = G^{[k+1]}$ then, clearly, $G^{[k]} = G^{[l]}$ for every $l \geq k$ and we say that the lower central series *terminates* at $G^{[k]}$. The lower central series is a series for G provided it terminates at $\{1\}$.

(E9.10) *A group is nilpotent if and only if the lower central series terminates at $\{1\}$. The nilpotency class of a nilpotent group G is equal to the length of the lower central series.*

Define a sequence of groups as follows:

$$Z_0 := \{1\};$$
$$Z_{i+1} = \{x \in G \mid \forall y \in G, [x, y] \in Z_i\}.$$

We obtain an ascending sequence of groups

$$\{1\} = Z_0 \trianglelefteq Z_1 \trianglelefteq Z_2 \trianglelefteq \cdots$$

which is called the *upper central series* of G. We say that this series *terminates* at Z^k if, for some k, $Z_k = Z_{k+1}$. The upper central series is a series for G provided it terminates at G. Note that $Z_1(G)$ is just the center of G; we refer to Z_i as the *i-th center* of G.

(E9.11) *Prove that, for all i, Z_{i+1}/Z_i is the center of G/Z_i. Deduce that a group is nilpotent if and only if the upper central series terminates at G. The nilpotency class of a nilpotent group G is equal to the length of the upper central series.*

(E9.12) *Prove that if a prime t divides the order of a finite nilpotent group G, then G has a unique Sylow t-subgroup. Deduce that G is the direct product of its Sylow subgroups.*

Write $F(G)$ for the largest normal nilpotent subgroup of G. We refer to $F(G)$ as the *Fitting subgroup of G*.

(E9.13) *Prove that if G is solvable, then $C_G(F(G)) = Z(F(G))$.*

9.5. **Iwasawa's Criterion.** In this section we give an illustration of how the notion of solvability can be used in studying simple groups. Specifically, we state a famous lemma of Iwasawa which gives a criterion for a finite permutation group to be simple. This lemma will be vital when we come to study the finite classical groups.

Lemma 9.3. (Iwasawa's criterion) *Let G be a finite group acting primitively on a set Ω. Let $\omega \in \Omega$ and assume that G_ω has a normal subgroup A which is abelian such that*

$$\langle A^g \mid g \in G \rangle = G$$

If $K \lhd G$, either $K \leq G_{(\Omega)}$ or $G' \leq K$. In particular if G is perfect and faithful on Ω, then G is simple.

(E9.14) *Use Iwasawa's criterion to show that A_5 is simple.*

(E9.15) *Now use Iwasawa's criterion to show that A_n is simple for $n \geq 5$. Hint: consider the action on unordered triples from $\{1, \ldots, n\}$.*

Proof. Let K be a normal subgroup of G that is not contained in $G_{(\Omega)}$. Lemma 5.2 implies, therefore, that K acts transitively on Ω and hence $G = G_\omega K$ (use the Orbit-Stabilizer Theorem to see this). Thus, for all $g \in G$, there exists $g_1 \in G_\omega, k \in K$ such that $g = g_1 k$ and this implies, in particular, that

$$\{A^g \mid g \in G\} = \{A^k \mid k \in K\}.$$

Now, since $\langle A^k \mid k \in K \rangle \leq AK \leq G$ we conclude that $G = AK$. Then

$$G/K = AK/K \cong A/A \cap K.$$

Since the right hand side is a quotient of an abelian group it must itself be abelian, and we conclude that G/K is abelian. Hence, by (E9.3), $K \geq G'$. □

(E9.16) *Prove the following variant on Iwasawa's criterion: Suppose that G is a finite perfect group acting faithfully and primitively on a set Ω, and suppose that the stabilizer of a point has a normal soluble subgroup S, whose conjugates generate G. Then G is simple.*