12. Linear groups acting on projective space

In this section V is an n-dimensional vector space over a field k. We investigate the action of $\mathrm{PGL}(V)$ (or, equivalently, of $\mathrm{GL}(V)$) on $\mathrm{PG}(V)$. You will see that we gradually put together the pieces to apply Iwasawa's criterion to $\mathrm{PSL}(V)$.

12.1. Transitivity properties.

Lemma 12.1. *If $n \geq 2$, then $\mathrm{PGL}(V)$ acts 2-transitively on the set of points of $\mathrm{PG}(V)$.*

Proof. Let (v_1, v_2) and (w_1, w_2) be pairs of linearly independent vectors in V. Extend both pairs to bases: $B_V := \{v_1, \dots, v_n\}$ and $B_W\{w_1, \dots, w_n\}$ Now let A be the matrix

$$((v_1)_W \, (v_2)_W \, \cdots \, (v_n)_W),$$

where we write $(v_i)_W$ to mean the column vector v_i written in terms of the basis B_W. Now it is easy to see that $v_i^T = w_i^T A^T$ for $1, \dots, n$ and we are done. □

In fact Lemma 12.1 can be deduced from a stronger result which we leave as an exercise. Recall that if a group acts transitively on a set with trivial stabilizers, then the action is called *regular* or *sharply transitive*.

> **(E12.1*)** *A tuple of $n+1$ points in $\mathrm{PG}_{n-1}(k)$ (i.e. a tuple of $n+1$ lines in V) is said to be* special *if no n of its entries lie in a hyperplane. Write Σ_V for the set of special tuples. Prove that the action of $\mathrm{PGL}(V)$ on Σ_V is regular.*
>
> **(E12.2*)** *Prove that $\mathrm{PSL}_n(k)$ is 2-transitive on the points of $\mathrm{PG}_{n-1}(k)$. Prove, furthermore, that $\mathrm{PSL}_n(k)$ is 3-transitive if and only if $n = 2$ and every element of k is a square.*

Lemma 12.2. *Let $G = \mathrm{SL}_n(k)$ and $\omega \in \Omega$, the set of points of $\mathrm{PG}(V)$. Then*

$$G_\omega \cong Q \rtimes GL_{n-1}(k)$$

where Q is an abelian group isomorphic to the additive group $(k^{n-1}, +)$.

The proof below sheds more light on the groups Q and G_ω.

Proof. Since G acts transitively on the set of points of $\mathrm{PG}(V)$, all stabilizers are isomorphic. We set $\{e_1, \dots, e_n\}$ to be the basis of elementary vectors and observe that the stabilizer of $\langle e_n \rangle$ is

$$G_{\langle e_n \rangle} = \left\{ g := \left(\begin{array}{ccc|c} & & & a_1 \\ & A & & \vdots \\ & & & a_{n-1} \\ \hline 0 & \cdots & 0 & a \end{array} \right) \,\middle|\, \begin{array}{l} a_1, \dots, a_{n-1} \in k, a \in k^*, \\ A \in \mathrm{GL}_{n-1}(k), \\ a = \frac{1}{\det(A)} \end{array} \right\}. \tag{11}$$

Now there is a natural epimomorphism

$$G_{\langle e_n \rangle} \to \mathrm{GL}_{n-1}(q), g \mapsto A,$$

and the kernel of this map is the group

$$Q := \left\{ g := \left(\begin{array}{ccc|c} & & & a_1 \\ & I & & \vdots \\ & & & a_{n-1} \\ \hline 0 & \cdots & 0 & 1 \end{array} \right) \mid a_1, \dots, a_{n-1} \in k \right\} \tag{12}$$

which is clearly isomorphic to $(k^{n-1}, +)$. Thus $G_{\langle e_n \rangle}$ is an extension of Q by $\mathrm{GL}_{n-1}(q)$.

> **(E12.3*)** *Prove that this extension is split.*

□

12.2. **Transvections and generation.** The motivation for this subsection is to establish that $\mathrm{SL}(V) = \langle Q^g \mid g \in \mathrm{SL}(V)\rangle$ where Q is the subgroup of $\mathrm{GL}(V)$ defined in Lemma 12.2.[37]

A *transvection* on V is an element $t \in \mathrm{GL}(V)$ such that

- $\mathrm{r}k(t-I) = 1$;
- $(t-I)^2 = 0$.

We define

- the *axis* of t to be $\ker(t-I)$;
- the *centre* of t to be $\mathrm{Im}(t-I)$.

Notice that the axis of t is a hyperplane in V, while the centre is a 1-dimensional subspace of that hyperplane.

Lemma 12.3. *All transvections lie in* $\mathrm{SL}_n(q)$ *and are conjugate in* $\mathrm{GL}_n(q)$.

Proof. We choose a basis for V as follows:

- $v_n \in \mathrm{Im}(t-I)$;
- $v_2, \dots, v_{n-1}$ are chosen so that $\langle v_1, \dots, v_{n-1}\rangle$ is the centre of t;
- v_1 such that $v_n = v_1^{t-I}$.

Then it is easy to see that the matrix of t with respect to this matrix is

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \tag{13}$$

(Recall that we are acting on the right.) The result follows. □

By comparing (12) and (13) it is clear that the group Q in Lemma 12.2 contains a transvection.

(E12.4*) *Prove that if $n \geq 3$, then $\mathrm{SL}_n(k)$ contains a unique conjugacy class of transvections. Prove that if $n = 2$, then $\mathrm{SL}_n(k)$ contains one or two conjugacy classes of transvections. Can you characterise when $\mathrm{SL}_n(k)$ contains two conjugacy classes, and describe how the subgroup Q intersects each class? (In particular you should show that each class has non-empty intersection with Q.)*

Lemma 12.4. *If $n \geq 2$, then $\mathrm{SL}_n(k)$ is generated by the set of all transvections.*

Proof. Let $G = \mathrm{SL}_n(k)$ and define

$$D := \langle t \mid t \text{ is a transvection}\rangle.$$

We proceed by induction on n.

Let $n = 2$ and consider the group

$$Q := \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in k \right\}.$$

(This is the same group Q which was defined in the proof of Lemma 12.2, and is normal in $G_{\langle e_2\rangle}$.) Observe that all non-identity elements of Q are transvections. Furthermore it is easy to see that Q is transitive on the set of 1-subspaces of V that are distinct from $\langle e_2\rangle$. Since we can easily find a transvection that does not fix $\langle e_2\rangle$, we conclude that D is 2-transitive on the points of $\mathrm{PG}_{n-1}(k)$.

Thus we will be done if we can show that the stabilizer in G of a pair of distinct points of $\mathrm{PG}_{n-1}(k)$ is equal to the stabilizer in D of that pair. One can calculate directly that

$$G_{(\langle e_1\rangle, \langle e_2\rangle)} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in k^* \right\}.$$

Now we can write elements of this group as products of transvections as follows:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a-1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a-a^2 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}. \tag{14}$$

[37] Refer to Lemma 9.3, Iwasawa's criterion, to see why we would want to know this.

We conclude that $\mathrm{SL}_2(k)$ is generated by transvections.

Now consider $n > 2$. Let v_1 and v_n be linearly independent vectors in V. We can extend to a basis $\{v_1, v_2, \dots, v_n\}$ and now observe that the matrix (13) is a transvection which maps $\langle v_1\rangle$ to $\langle v_1 + v_n\rangle$. One quickly concludes that D is transitive on points of $\mathrm{PG}_{n-1}(k)$. It is sufficient, then, to show that $G_{\langle v_n\rangle} = D_{\langle v_n\rangle}$. We will do this by appealing to induction.

Recall first that the structure of $G_{\langle v_n\rangle}$ is given in Lemma 12.2. Next observe that $G_{\langle v_n\rangle}$ acts naturally on the quotient space $V/\langle v_n\rangle$ and, moreover, that transvections in $G_{\langle v_n\rangle}$ induce transvections (or the identity) on $V/\langle v_n\rangle$Thus, by induction, they generate the group $\mathrm{SL}_{n-1}(k)$ on this quotient space. Thus if $g \in G_{\langle v_n\rangle}$ has the form (11) then, by multiplying g by transvections we obtain an element

$$h := \begin{pmatrix} a^{-1} & 0 & \cdots & 0 & a_1 \\ 0 & 1 & & \vdots & a_2 \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & a_{n-1} \\ 0 & \cdots & \cdots & 0 & a \end{pmatrix}$$

where $a, a_1, \dots, a_{n-1} \in k$ with $a \neq 0$. Further multiplication by transvections allows us to assume that $a_1 = \cdots a_{n-1} = 0$, and now the identity (14) allows us to multiply by more transvections to assume that $a = 1$. Thus we have written g as a product of transvections and the result follows. □

Lemma 12.5. *Let t be a transvection in $\mathrm{SL}_n(k)$. Then t is a commutator except when $n = 2$ and $|k| \leq 3$.*

Proof. If $n = 1$, $\mathrm{SL}_n(k)$ is trivial and the result is immediate, so assume that $n \geq 2$.

By (E12.4) all transvections are conjugate to a non-trivial element from Q, thus we need only show that all non-trivial elements of Q are commutators. If $n = 2$ and $|k| > 3$ one can do this by taking $a, x \in k$ with $a^2 \neq 0, 1$ and observing that

$$\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & (a^2-1)x \\ 0 & 1 \end{pmatrix}. \tag{15}$$

Clearly, as x varies across k, we cover all non-trivial elements of Q.

If $n > 2$, then all transvections are conjugate, and we need only show that the transvection (13) is a commutator. If $|k| > 3$, then this is achieved using (15) by taking $x = \frac{1}{a^2-1}$, and enlarging each matrix to size n by n, by appending blocks equal to the identity of rank $n - 2$.

We are left with the case $n > 2$ and $|k| = 2$ or 3.

(E12.5*) *Prove the remaining case.*

□

The two previous lemmas imply the following corollary.

Corollary 12.6. $\mathrm{SL}_n(k)$ *is perfect except when $n = 2$ and $q \leq 3$.*

Remark. The projective image of a transvection in $\mathrm{PSL}_n(k)$ is called an *elation*. Now it is easy to see that the three previous results, Lemmas 12.4 and 12.5 and Corollary 12.6 all remain true if one replaces all instances of the word 'transvection' by 'elation' and all instances of '$\mathrm{SL}_n(k)$' by '$\mathrm{PSL}_n(k)$'.

12.3. **Finite groups.** In this section $k = \mathbb{F}_q$. In this situation, for

$$\mathrm{X} \in \{\Gamma, \mathrm{G}, \mathrm{S}, \mathrm{P}\Gamma, \mathrm{PG}, \mathrm{PS}\},$$

we write $\mathrm{XL}_n(q)$ as a synonym for $\mathrm{XL}_n(k)$.

Theorem 12.7. *The group $\mathrm{PSL}_n(q)$ is simple except when $n = 2$ and $q \leq 3$.*

Proof. If $n = 1$, then $\mathrm{PSL}_n(q)$ is trivial and the result is immediate. Assume that $n \geq 2$ and that $q > 3$. Observe that (E12.2) implies that $\mathrm{PSL}_n(q)$ acts faithfully and primitively on the set of points of $\mathrm{PG}_{n-1}(q)$.

Let Q_0 be the subgroup of $\mathrm{PSL}_n(q)$ equal to the projective image of the subgroup Q in Lemma 12.2. Now (E12.4) implies that $\bigcup_{g\in G} Q^g$ contains all transvections of $\mathrm{SL}_n(q)$ and Lemma 12.4 implies, therefore, that

$$\langle Q_0^g \mid g \in G\rangle = \mathrm{PSL}_n(q).$$

Finally Corollary 12.6 implies that $\mathrm{PSL}_n(q)$ is perfect. Now Lemma 9.3 (Iwasawa's Criterion) implies that $\mathrm{PSL}_n(q)$ is simple. □

In the next result and hereafter, for integers k, l, we write (k, l) for their greatest common divisor.

Proposition 12.8.

$$\begin{aligned}|\mathrm{GL}_n(q)| &= (q^n-1)(q^n-q)\cdots(q^n-q^{n-2})(q^n-q^{n-1})\\ &= q^{\frac{1}{2}n(n-1)}(q^n-1)(q^{n-1}-1)\cdots(q^2-1)(q-1).\\ |\mathrm{PGL}_n(q)| &= |\mathrm{SL}_n(q)| = |\mathrm{GL}_n(q)|/(q-1).\\ |\mathrm{PSL}_n(q)| &= |\mathrm{SL}_n(q)|/(q-1,n).\end{aligned}$$

Proof. We write down elements of $\mathrm{GL}_n(q)$ with respect to a fixed basis for $V = \mathbb{F}_q^n$. There are $q^n - 1$ choices for the first column, then $q^n - q$ choices for the second column (since we cannot choose vectors that are in the span of the first), then $q^n - q^2$ choices for the third column, and so on. The identity for $|\mathrm{GL}_n(q)|$ follows.

Now $|\mathrm{PGL}_n(q)| = |\mathrm{GL}_n(q)|/|K|$ where, by (E11.8),

$$K = \{\alpha I \in \mathrm{GL}(V) \mid \alpha \in k\}.$$

Since $|K| = q - 1$, the identity for $|\mathrm{PGL}_n(q)|$ follows. On the other hand $|\mathrm{SL}_n(q)|$ is the kernel of the determinant map $\mathrm{GL}_n(q) \to k^*$. Since this map is surjective, the first isomorphism theorem for groups yields the identity for $\mathrm{SL}_n(q)$.

Finally observe that $|\mathrm{PSL}_n(q)| = |\mathrm{SL}_n(q)|/|K \cap \mathrm{SL}_n(q)|$. Using the fact that k^* is cyclic of order $q-1$ we conclude immediately that $|K \cap \mathrm{SL}_n(q)| = (n, q-1)$ and we are done. □

(E12.6*) *Show that the set of upper-triangular matrices with 1's on the diagonal is a Sylow p-subgroup of* $\mathrm{GL}_n(q)$.

(E12.7)

1. *Write down elements of order* 3, 4 *and* 5 *in the group* $\mathrm{SL}_2(5)$.
2. *Write down elements of order* 6, 7 *and* 8 *in the group* $\mathrm{SL}_2(7)$.
3. *(Harder). Can you write down elements of order* $q-1$, p *and* $q+1$ *in the group* $\mathrm{SL}_2(q)$*? Can you describe the structure of a Sylow* t*-subgroup of* $\mathrm{SL}_2(q)$ *for different* t*?*

(E12.8) *What are the orders of elements in* $\mathrm{SL}_3(q)$*?*

(E12.9) *Describe the conjugacy classes of* $\mathrm{PGL}_2(q)$*. Ascertain which of these classes lies in* $\mathrm{PSL}_2(q)$ *and list those that 'split' into more than one* $\mathrm{PSL}_2(q)$*-conjugacy class. Do similarly for* $\mathrm{PGL}_3(q)$.

Isomorphisms between 'different' simple groups turn out to be very significant in the group theory universe. The next result discusses some of these.

Proposition 12.9.
1. $\mathrm{SL}_2(2) \cong S_3$;
2. $\mathrm{PSL}_2(3) \cong A_4$;
3. $\mathrm{SL}_2(4) \cong PSL_2(5) \cong A_5$;
4. $\mathrm{PSL}_2(7) \cong \mathrm{SL}_3(2)$;
5. $\mathrm{PSL}_2(9) \cong A_6$;
6. $\mathrm{SL}_4(2) \cong A_8$.

Note that we write SL rather than PSL in cases where $(n, q-1) = 1$, since in these cases $\mathrm{PSL}_n(q) \cong \mathrm{SL}_n(q)$.

Proof of (1) to (4) only. A 2-dimensional vector space over $\mathbb{F}_2$ has 3 lines through the origin, on which $\mathrm{SL}_2(2)$ acts faithfully. Thus $\mathrm{SL}_2(2)$ embeds into S_3; comparing orders we conclude that $\mathrm{SL}_2(2) = S_3$.

A 2-dimensional vector space over $\mathbb{F}_3$ has 4 lines through the origin, on which $\mathrm{PSL}_2(3)$ acts faithfully. Thus $\mathrm{PSL}_2(3)$ embeds into S_4 as a subgroup of index 2. Now S_4 has a unique subgroup of index 2, namely A_4, thus $\mathrm{PSL}_2(3) \cong A_4$.

A 2-dimensional vector space over $\mathbb{F}_4$ has 5 lines through the origin, on which $\mathrm{SL}_2(4)$ acts faithfully. Thus $\mathrm{SL}_2(4)$ embeds into S_5 as a subgroup of index 2. Either $\mathrm{SL}_2(4) = A_5$ or $|A_5 : \mathrm{SL}_2(4) \cap A_5| = 2$. But, since A_5 is simple and index 2 subgroups are normal, the latter possibility cannot occur. Thus $\mathrm{SL}_2(4) = A_5$.

We use the same reasoning on the simplicity of A_5 to see that if $\mathrm{PSL}_2(5)$ acts on a set of size 5, then $\mathrm{PSL}_2(5)$ is isomorphic to A_5. We claim that $\mathrm{PSL}_2(5)$ has 5 Sylow 2-subgroups. One can compute these directly or else observe that the possible number of Sylow 2-subgroups is 1,3,5 or 15. Since $\mathrm{PSL}_2(5)$ is simple the first two are ruled out (why?). Now observe that, since a Sylow 2-subgroup of $\mathrm{PSL}_2(5)$ is equal to the centralizer of all of its non-trivial elements, we conclude that these elements lie in a unique Sylow 2-subgroup. If there were 15 Sylow 2-subgroups, then we would have 45 elements of order 2 in $\mathrm{PSL}_2(5)$ which is impossible - there are, for instance, 24 elements of order 5.

We know that $\mathrm{Aut}(\mathrm{PG}_2(2)) = \mathrm{SL}_3(2)$ and, by (E11.2), we know that $\mathrm{PG}_2(2)$ is equal to the incidence structure represented in Figure 4, the Fano plane. Thus it is sufficient to show that $\mathrm{PSL}_2(7)$ acts non-trivially on the Fano plane – since $\mathrm{PSL}_2(7)$ is simple, this action will therefore be faithful, inducing an embedding of $\mathrm{PSL}_2(7)$ into $\mathrm{SL}_3(2)$ which must be an isomorphism by comparison of orders.

Now define an incidence structure $\mathcal{I}$ as follows: Let S be a Sylow 2-subgroup of $G = \mathrm{PSL}_2(7)$ - it is dihedral of order 8 and contains two K_4-subgroups, U and V. One can check that $N_G(U) \cong N_G(W) \cong S_4$, thus there are 7 conjugates of U and 7 conjugates of V; what is more these conjugates are distinct. We set the conjugates of U to be the points of our incidence structure, the conjugates of V to be the lines, and say that a point and a line are incident if they are contained in the same Sylow 2-subgroup of G. Now one must check that this incidence structure is isomorphic to the Fano plane, and that the natural conjugation action of G on the conjugates of U and V respectively, induces an action on $\mathcal{I}$.

(E12.10*) *Check the details of the last paragraph.*

□

It turns out that the above list is a complete list of all isomorphisms between groups of form $\mathrm{PSL}_n(q)$ and A_n and S_n (a hardish fact that we won't prove). In fact this list contains almost all instances of a coincidence of cardinality between groups of form $\mathrm{PSL}_n(q)$ and groups of form A_n - there is one more such coincidence which is considered in the next exercise.

(E12.11*) *Prove that* $\mathrm{PSL}_3(4) \not\cong \mathrm{SL}_4(2) \cong A_8$, *despite the fact that these groups have the same orders.*

We have seen, in Lemma 3.5, that $\mathrm{PSL}_n(q)$ embeds into its own automorphism group. In fact, as the next proposition makes clear, we have (kind of) already seen the automorphism group of $\mathrm{PSL}_n(q)$. To state the proposition we need one definition: fix a basis of $V = \mathbb{F}_q^n$ and define

$$\iota : \mathrm{PSL}_n(q) \to \mathrm{PSL}_n(q), x \mapsto x^{-T}.$$

To be clear: given $x \in \mathrm{PSL}_n(q)$, let X be an element in $\mathrm{SL}_n(q)$ that projects onto x, then define x^ι to be the projective image of X^{-T}, the inverse transpose of the matrix X with respect to the fixed basis.

(E12.12) *Check that this is a well-defined automorphism of* $\mathrm{PSL}_n(q)$.

Proposition 12.10. $\mathrm{Aut}(\mathrm{PSL}_n(q)) = \begin{cases} \mathrm{P\Gamma L}_2(q), & \text{if } n = 2; \\ \mathrm{P\Gamma L}_2(q) \rtimes \langle \iota \rangle, & \text{if } n \neq 3. \end{cases}$

The proof is omitted, although one inclusion is covered in the following exercise. You should compare the statement of the proposition to the statement of (E11.16).

(E12.13*) *Prove that*

$$\mathrm{Aut}(\mathrm{PSL}_n(q)) \geq \begin{cases} \mathrm{P\Gamma L}_2(q), & \text{if } n = 2; \\ \mathrm{P\Gamma L}_2(q) \rtimes \langle \iota \rangle, & \text{if } n \neq 3. \end{cases}$$

Hint: you need to study the natural action of, say, $\mathrm{P\Gamma L}_n(q)$ *on its normal subgroup* $\mathrm{PSL}_n(q)$.