RESEARCH PROPOSAL

1. Background

The primary goal of the proposed research is a proof of the Product Decomposition Conjecture (PDC). This conjecture was first stated by Liebeck, Nikolov and Shalev [12]:

Product Decomposition Conjecture. *There exists an absolute constant c such that if G is a finite simple group and S is a subset of G of size at least two, then G is a product of N conjugates of S for some $N \leq c \log |G| / \log |S|$.*

1.1. **Context.** The study of the PDC lies within the domain of finite group theory, albeit with multiple connections to other areas of mathematics, including arithmetic combinatorics, permutation group theory and number theory.

The statement of the PDC can be thought of as a statement about the *width* of a group: given a group G and a set A, the *width of G with respect to A*, denoted by $w(G, A)$, is defined to be the smallest integer N such that G can be written as a product of N conjugates of A. If no such integer exists, then we say that $w(G, A)$ is undefined.

Observe that if G is finite and simple, and $|A| \geq 2$, then $w(G, A)$ is always defined. Now the PDC asserts that, in this case, the width of G with respect to A is as small as it could possibly be. One sees this by observing that the integer N must be greater than or equal to $\log |G| \big/ \log |S|$ by order considerations.

Upper bounds on width have been studied by many authors in a wide variety of specific situations. A full proof of the PDC would generalize all of the qualitative results known for finite simple groups and would represent a dramatic advance in our understanding of this subject.

1.2. **Past and current work.** A weaker version of the PDC, again due to Liebeck, Nikolov and Shalev [13], was extant in the literature prior to the formulation of the PDC:

Conjecture 1. *There exists an absolute constant c such that if G is a finite simple group and H is any nontrivial subgroup of G, then G is a product of N conjugates of H for some $N \leq c \frac{\log |G|}{\log |H|}$.*

Conjecture 1 itself represents a dramatic generalization of a host of earlier work on product decompositions of finite simple groups, most of which prove Conjecture 1 for particular subgroups H. For instance, in [15] it is proved that a finite simple group of Lie type in characteristic p is a product of 25 Sylow p-subgroups (see also [1] for a recent improvement from 25 to 5).

Further positive evidence for Conjecture 1 is provided by [14], [17] and [18] (when H is of type SL_n). Certain results of this type are essential to prove that finite simple groups can be made into expanders (see the announcement [10]).

All of the results just mentioned apply not only to Conjecture 1 but to its generalization, the PDC. In addition to these, there are two important results in the literature that explicitly address particular cases of the PDC, and are not limited to the subgroup context. The first is a celebrated theorem of Liebeck and Shalev [16] that predated the PDC and directly influenced its formulation:

Theorem 2. *There exists an absolute positive constant a such that, if G is a finite simple group and S is a nontrivial normal subset of G, then $G = S^m$, where $m \leq a \frac{\log |G|}{\log |S|}$.*

Theorem 2 has been widely applied to a number of problems in algebra involving, especially, *mixing* [11], *group factorizations* and the analysis of *word maps* [20].

The second important result is due to the PI, Pyber, Short and Szabó [7]. Their result proves a rank-dependent version of the PDC for finite groups of Lie type:

Theorem 3. *Fix a positive integer r. There exists a constant $c = c(r)$ such that if G is a finite simple group of Lie type of rank r and S is a subset of G of size at least two then G is a product of N conjugates of S for some $N \leq c \log |G| / \log |S|$.*

In light of the Classification of Finite Simple Groups, Theorem 3 proves the PDC for all families of simple groups except (a) the alternating groups, and (b) the classical groups of unbounded rank. In particular Theorem 3 yields the PDC for the exceptional groups.

The primary focus of the proposed research is the lacuna (b); that is, we would like to prove the Product Decomposition Conjecture for the classical groups of unbounded rank. Some of our methods are also expected to address lacuna (a), however the behaviour of the alternating groups is in general quite different to the groups of Lie type so this line of inquiry is a little more speculative.

1.3. **The wider literature.** The idea of width is connected to a similar notion, that of *diameter*: given a group G and a generating set $A \subseteq G$, we define the *diameter of G with respect to A*, $\mathrm{diam}(G, A)$, to be the smallest integer N such that $G = A^N$. If we restrict ourselves to the situation where A contains the identity and is symmetric (i.e. $A = A^{-1}$), then it is clear that the diameter of G with respect to A is simply the diameter of the Cayley graph $\Gamma(G, A)$.

Just as for width, it is natural to seek upper bounds for the diameter of a group and in this context the analogue of the PDC is Babai's Conjecture, a celebrated and much-studied conjecture in group theory. Indeed this conjecture has recently been partially proven thanks to breakthrough work by Helfgott [9] and others on *growth* in the finite groups of Lie type.

The definitive statement concerning growth in simple groups is the following Product Theorem, due separately to Breuillard, Green and Tao [4] and Pyber and Szabó [19]:

Product Theorem. *Fix a positive integer r. There exists a positive constant $\eta = \eta(r)$ such that, for G a finite simple group of Lie type of rank r and S a generating set of G, either $S^3 = G$ or $|S^3| \geq |S|^{1+\eta}$.*

The lower bound given in the Product Theorem yields, by simple iteration, a bound of the form $\mathrm{diam}(G, A) \leq (\log |G|)^C$ for a constant C depending only on r. This is a rank-dependent version of Babai's conjecture.

A key point here is the rank-dependency of the constant η in the Product Theorem - this is not a vagary of the proof; examples are known that demonstrate that η cannot be absolute.

In contrast, when one is allowed to take *conjugates* of a set (as we do when considering width), it is thought that the situation is very different [7]:

Conjecture 4. *There exists $\varepsilon > 0$ and a positive integer b such that if S is a subset of a finite simple group G then for some g in G we have $|SS^g| \geq |S|^{1+\varepsilon}$ or G is the product of b conjugates of S.*

The resemblance between Conjecture 4 and the Product Theorem should be clear, but it is the differences that we wish to highlight. First, in the conjecture one obtains a lower bound using only **two** conjugates of S, rather than the three needed in the Product Theorem. This phenomenon is explained by the Skew Doubling Lemma, discussed briefly below.

Second, note the intriguing possibility of the existence of an absolute contant, ε, for products of conjugates. The existence of a constant would be of great interest in itself; in addition, by iteration just as before, one would immediately obtain $w(G, A) \leq C(\log |G| / \log |A|)^D$ for absolute constants C and D. This is a polynomial version of the PDC.

2. Academic Impact

The Product Decomposition Conjecture is a powerful statement about the structure of the finite simple groups. Given the ubiquity of these objects in group theory, combinatorics and elsewhere, a proof of the PDC would be of international interest. (This is discussed more in the final section of this document.)

We have mentioned already a number of direct areas of application of the PDC: these include mixing and expansion (in the areas of probabilistic group theory, combinatorics and theoretical computer science through the study of efficient networks), sieving (in number theory) and the study of word maps (in group theory).

We limit ourselves to discussing just one of these applications, namely *expansion*: the idea here is to explicitly construct expander families, a mathematical model of an efficient network. Such families have been known to exist by probabilistic arguments but, until recently, explicit constructions have been hard to come by. However, using the ideas of growth mentioned above (specifically Helfgott's breakthrough [9]), Bourgain and Gamburd were able to construct expander families using Cayley graphs of the groups $\mathrm{SL}_2(p)$ [2].

This breakthrough was then exploited by various authors to construct other expander families using Cayley graphs of other families of finite simple groups. The methodology used was of two types: the first involved adapting the arguments of Bourgain and Gamburd directly for use with more general growth results such as the Product Theorem. The second method is relevant to the PDC: here authors constructed expander families for a family of groups G_n by proving results on width with respect to certain subgroups H_n that were already known to yield expanders. It turns out that upper bounds on width (*à la* the PDC) are exactly what is needed to prove expansion of a group given expansion of a subgroup.

In addition to applications of the PDC, we note that this particular type of question has proven worth as a *signpost* for interesting and unexpected connections between different areas of mathematics. By way of illustration we go back to Theorem 3, the rank-dependent version of the PDC.

In order to prove Theorem 3 new light was shone on some classical results of additive combinatorics. Specifically, a new proof by Petridis of the classical Doubling Lemma due to Ruzsa was adapted to a much more general setting. The original Doubling Lemma concerned subsets of abelian groups while the generalization (referred to as the 'Skew Doubling Lemma') applied to normal subsets of general groups.

This development had at least two interesting consequences: firstly, it pointed the way to other similar generalizations of results from classical additive combinatorics to the non-abelian setting (in particular, Plünnecke's Theorem). Secondly, it yielded a partial proof of a conjecture of Shalev concerning the growth of conjugacy classes [20]:

Conjecture 5. *There exists an absolute constant $\varepsilon > 0$ such that for every finite simple group G and every conjugacy class C of G have*

$$|C^2| \geq \min\{|C|^{1+\varepsilon}, |G| - \delta\}$$

where $\delta = 1$ if C is real, and $\delta = 0$ otherwise.

The Skew Doubling Lemma, combined with Theorem 2 yields the result for all 'small' conjugacy classes in G. To prove the full conjecture requires a result for conjugacy classes C of order at least $|G|^\delta$ (for some constant δ); the study of such conjugacy classes is discussed in §4 below. Conjecture 5 is a natural counterpart to the PDC; its importance is illustrated by the fact that a proof would imply the famous conjecture of Thompson up to finitely many exceptions.

2.1. **Collaboration.** The principle collaborative partnership in this project is with Professor Laszlo Pyber of the Rényi Institute. Professor Pyber is one of the world's foremost algebraists and, as can be seen by a brief perusal of the references to this document, he has already made seminal contributions to the study of the width of groups and related applications. Indeed, the strongest statement concerning the PDC, Theorem 3, is joint work of the PI and Professor Pyber, in collaboration with two other mathematicians.

Since one of our main strategies (see §4) will be to revisit the proof of Theorem 3, Professor Pyber is in the best possible position in terms of current knowledge. What is more, it is expected that the primary obstacles to generalizing this proof will concern technical questions about finite simple groups. Professor Pyber's unquestioned mastery of the technical theory of the finite simple groups will be of crucial importance.

In addition to work with Pyber, the PI expects to have significant discussions with Professor Harald Helfgott, of CNRS, Paris. Professor Helfgott is a mathematician of international renown; indeed it is his work on $\mathrm{SL}_2(p)$ which opened the way to the study of growth in non-abelian groups, leading ultimately to the Product Theorem amongst other important results. (See the *Track Record* document for detail concerning past high-level collaboration between the PI and Helfgott.)

Finally, the PI will discuss and disseminate the research to a wide range of world experts with a view to potential collaboration. These include, but are not limited to, Dr Ian Short (Open University), Dr Pablo Spiga (Milano-Bicocca) and Professor Cheryl Praeger (Western Australia).

3. Research Hypothesis and Objectives

Our first aim is to prove the PDC. The timeliness of such a study is hopefully evident in the publication list - this is a question that has received a great deal of attention from a variety of authors, with publications in the very best mathematics journals in the world.

The novelty of the proposed approach is in its combination of a known successful strategy (revisiting the proof of Theorem 3) combined with a deep understanding of the theory of the finite simple groups, as well as a new and significant connection to the classical study of bases (see §4.2).

The research sets ambitious goals, but along the way there are a number of measurable objectives which themselves are of interest to researchers in the field. These objectives are listed in detail in the *Work Plan* document, to which we refer the reader; they include Conjecture 1 and 4 above.

Although the PDC, itself, is certainly an ambitious goal, preliminary investigations suggest that a number of these more modest objectives are very much within the range of current techniques. Their achievement would, nevertheless, represent a very significant development in the study of width in groups.

4. Programme and Methodology

The proposed research is of a pure mathematical nature, so will proceed through the careful study of known proofs of related results and the development of technical theory.

In order to be as explicit as possible we outline below the programme for proving the PDC for the family $\mathrm{PSL}_n(q)$; our analysis of the remaining classical groups will proceed along the same lines. Much of what we describe can also be applied to the alternating groups, however the situation there has important technical differences that are likely to require more leaps of inspiration.

4.1. **Overhauling the proof of Theorem 3.** We start with a group $G = \mathrm{PSL}_n(q)$ and a subset A of size 2. Recall that Theorem 3 yields a rank-dependent version of the PDC. The proof of this theorem differs according to the size of A, and our analysis will be similarly split. Recall that $|G| \sim q^{n^2-1}$.

Suppose that A **is 'small'** ($|A| \lesssim q^n$). Existing analysis uses so-called 'greedy lemmas'. These are simple counting arguments that yield Conjecture 1 for very small subsets inside a finite simple group. Their range of usefulness is determined by the size of small conjugacy classes inside the given group. In the proof of Theorem 3 this range was determined in a relatively crude way; initial investigations strongly suggest that these methods may be extended to apply to larger ranges by means of a more careful analysis of the conjugacy structure of the classical groups.

Ultimately we are confident that we can prove results of the following form: *For every $c > 0$ there exists $\varepsilon > 0$ such that if A is a subset of a classical group G of rank r and level q with $|A| < q^{cr}$ then there is $g \in G$ such that $|A \cdot A^g| \geq |A|^{1+\varepsilon}$.* This statement constitutes Conjecture 4 for 'small sets'.

Suppose that A is 'middle sized' ($q^n \lesssim |A| \lesssim q^{n^2-n}$). In this case the proof of Theorem 3 made use of the Product Theorem, a famous result in the theory of growth (see §1). In this context, the Product Theorem is both a vital tool and the key weakness in the proof of Theorem 3. Its use mandates rank-dependency in the statement of the theorem – it is this rank-dependency that stops Theorem 3 being a full proof of the PDC for the simple groups of Lie type.

In the proposed research we will seek to use other tools – a natural candidate is Theorem 2. This is suggested by the fact that the greedy lemmas apply directly to sets A that do not have large intersection with conjugacy classes of G, while Theorem 2 has force exactly when A contains a conjugacy class of G. In order to bridge the gap between those two situations we need to prove a statement of the following kind:

There exists $\delta > 0$ and a positive integer b such that if A is a subset of a conjugacy class C with $|A| > |C|^\delta$, then there exist b conjugates of A whose product contains C.

Suppose that A is 'large' ($|A| \gtrsim q^{n^2-n}$). Here the Product Theorem also applies and also gives rank-dependency. In this context however we have other notions that naturally play a role, specifically that of *quasirandomness* due (at least in the group theory setting) to Gowers [8]. Gowers' ideas have already been generalized by the PI to the setting of group actions [6]. In order to apply to the PDC, one would hope to use the combinatorial framework provided by quasirandomness, but replace Gowers' original use of representation degree-bounds with an approach to the bounds on convolution that does not depend on rank.

In this setting we seek a statement of the following form: *There exists $\delta > 0$ and a positive integer b such that if C is a conjugacy class in a finite simple group G of size at least $|C|^\delta$, then $C^b = G$.* Such a statement can be seen as a weaker form of Conjecture 5.

4.2. **Connections with base size.** Given a permutation group G, a *base* is a set of points whose point-wise stabilizer is trivial. This is a classically studied group-theoretic quantity which has received a great deal of attention over many years.

The PI has recently uncovered connections between width and base-size. Preliminary research suggests that one can use known bounds on base size due to various authors including, in particular, Burness [5] to prove a version of Conjecture 4 for subgroups of a finite simple group G. For instance when $G = \mathrm{PSL}_n(q)$ it would appear that the following is true:

Let $\varepsilon = \frac{1}{2^{10}-1}$ and $b = 30$. If $H < G = \mathrm{PSL}_n(q)$ with $(n, q) \notin \{(2, 2), (2, 3)\}$, then for some $g \in G$ we have $|H.H^g| \geq |H|^{1+\varepsilon}$ or G is the product of b conjugates of H.

A first task in the proposed research will be to write a proof of the statement just given. The remaining classical groups will require some technical finesse as the results on base size in this setting are slightly more complicated. Nonetheless such a result would appear very much within reach of current techniques.

The notion of base-size can also be applied to subsets but this is as yet not properly understood. Such an extension will be a natural first point of study. The great advantage of the base-size approach, as can be seen in the statement above, is that it yields explicit values for the quantities ε and b.

5. National Importance

The proposed research fits within the *Mathematical Sciences* section of the EPSRC portfolio and, within this, in the *Algebra* theme. Moreover there are strong connections to the themes of *Logic and combinatorics* (see the earlier discussion of expander graphs), and to the theme of *Number Theory* (see the earlier discussion of additive combinatorics; we also mention applications to sieving which we have not discussed but which are well-established [3]).

The study of growth and width in finite groups is an active area of research with leading researchers around the world. With reference to width we mention Nikolay Nikolov (Oxford), Martin Liebeck (Imperial) and Aner Shalev (Hebrew University of Jerusalem) to whom the statement of the PDC is due. The study of growth has attracted the interest of some of the biggest names in mathematics, including Fields medallists Bourgain, Gowers, Lindenstrauss and Tao, as well as famous individuals such as Gamburd (CUNY), Green (Oxford), Helfgott (CNRS, Paris), Pyber (Rényi Institute) Sarnak (Princeton), and many others. Finally the study of base-size has been a long-standing subject of interest within the group theory community and has attracted the attention of Burness (Bristol), Guralnick (USC), Saxl (Cambridge) and many others.

It is quite clear that the study of width, growth, and related applications is a 'hot topic of modern mathematics, and the proposed research is a timely and important contribution which will undoubtedly receive international attention.

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TRACK RECORD

The proposed research area is in the area of finite group theory. The main focus of the research, the Product Decomposition Conjecture (PDC), concerns the *width* of a finite simple group. The PDC is the definitive statement concerning width in this context and a full proof of it would open a new chapter in the study of expansion in finite simple groups.

The PI completed his PhD at the University of Cambridge in 2005. He has held postdoctoral positions at IMSc, Chennai (India), at Bristol University (under Harald Helfgott) and at the Open University. He spent 2014 as a visiting professor at the University of Costa Rica and was appointed as a lecturer by the University of South Wales in January 2015.

The PI's research expertise is in the area of finite group theory; in particular, he has made significant contributions to the study of growth, width and applications thereof. Several of these were completed during his time in Bristol as a postdoctoral assistant to Helfgott; several more have been completed since that time (some as a co-author with Helfgott). He has also published a wide range of articles pertaining to the structure of the finite simple groups.

In addition to discussions with Helfgott, the proposed research will include discussions with Professor László Pyber of the Rényi Institute, Budapest. As detailed below, the PI has a track-record of collaboration with both Helfgott and Pyber with the results being published in journals of international renown.

Results on width. One of the most general statements concerning the PDC extant in the literature at this point is due to the PI , Pyber, Short and Szabó [8]. This result, a 'rank-dependent version of the PDC', forms the basis for one strand of the proposed research project. The PI was invited to speak about this result at an international conference in Banff, Canada in 2012.

In order to prove the PDC for large sets it is expected that, by analogy with the Product Theorem for growth, ideas connected to *quasirandomness* will need to be studied. The role of quasirandomness in this area was first understood by Gowers who introduced the idea of a *quasirandom group*. The PI extended this idea to the study of *quasirandom group actions* (so Gower's original notion is now understood to refer to the specific situation of a group acting on itself) [2].

Results on growth. A key breakthrough in the study of growth in non-abelian groups was a famous result of Helfgott concerning the groups $\mathrm{SL}_2(p)$ [9]. In the push to generalize this work to arbitrary groups of Lie type, the PI and Helfgott published the first growth result for groups of unbounded rank [6]. This generalization foreshadowed the later Product Theorem, the definitive statement concerning generating sets of finite groups of Lie type.

In the aftermath of the Product Theorem, study in growth focused on two separate conjectures, both going under the name 'the Helfgott-Lindenstrauss Conjecture', one of which has since been proven by Breuillard, Green and Tao. The second remains open in full generality, but it has been proven in a restricted setting by the PI and Helfgott [5]; this result remains the strongest statement in this setting to date. The PI was invited to speak about this result at an international conference organised by Professor Green at Gregynog Hall, Wales in 2012 as well as at workshops at Royal Holloway, London and ENS, Paris.

Applications of growth and width. The study of growth and width has been particularly intense due to the large number of applications that have been found for results in these areas. The PI has contributed to such applications.

In [4], the PI recast an important conjecture in graph theory, the Weiss conjecture, in terms of a 'spectral gap' statement for the associated adjacency matrix. This recasting made use of the notion of quasirandomness and connects the Weiss conjecture directly to the sort of expansion results that Bourgain and Gamburd have pioneered using Helfgott's results on growth.

In [7], the PI, along with Helfgott and Rudnev, generalized some of the foundation results of additive combinatorics using geometric properties of finite projective planes. Indeed, this paper generalizes the whole notion of growth from being an algebraic property of structures such as groups or fields to a geometric property of incidence systems.

Finally, in [1], the PI and five other authors gave partial results for Babai's conjecture on the diameter of Cayley graphs. These results are based on a new graph-theoretic study of the fixed points of multiply-transitive group actions.

Expertise in use of the Classification of Finite Simple Groups. The proposed research is likely to involve a careful study of the properties of the finite simple groups, a study that the PI is well placed to carry out. In addition to much of the above work (especially [1, 5, 6, 8]) which depended on deep knowledge of the CFSG, the PI has made use of the CFSG, combined with knowledge of the intricate structure of the finite simple groups, to answer questions in a variety of areas of mathematics. We highlight one example.

In a series of three papers, culminating in [3], the PI has used the CFSG to study those finite projective planes that admit an automorphism group that is transitive on the set of points. A 60-year-old conjecture of Ostrom and Wagner asserts that all such planes are known, and the work of the PI is the strongest statement about this conjecture in the literature to date. This result asserts that any plane that is a counterexample to the conjecture has either a solvable automorphism group, or else one whose only insoluble section is isomorphic to A_5.

The University of South Wales. The PI is a new member of the mathematical department at the University of South Wales. There is significant expertise in the department on codes and coding theory, with researchers including Emeritus Professor Derek Smith and head of department Dr. Stephanie Perkins. Coding theory has close connections to expander graphs, and so it is expected that the expertise of the department can be exploited in the application of the proposed research. As described in the Pathways to Impact document, the PI is undertaking a programme of talks seeking to communicate the proposed research and to promote collaboration within this department and other departments in the region.

The university has agreed to fund one PhD student, specifically to work with the PI on the proposed research. They are also supportive of plans to supervise a PhD student from the University of Costa Rica. This will be of great assistance to the PI, as well as helping to maximise impact.

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