RESEARCH STATEMENT

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Abstract. My research is in mathematical logic, primarily in set theory. One theme of my research is to turn established concepts relating to elementary embeddings backwards. So far, my research has focused on four topics: inverse limits of elementary embeddings between models of set theory, the large cardinals between supercompact and almost-huge, generalizations of the Kunen inconsistency, and elementary epimorphisms between models of set theory.

1. Introduction

I conduct research in mathematical logic. My primary research concerns set theory, in particular topics broadly related to large cardinals and forcing. My results of which I am proudest, and which I feel involved the most creativity, are from my work on inverse limits of elementary embeddings between models of set theory. This work takes the concept of the direct limit of a system of elementary embeddings, a concept which is ubiquitous in set theory, and turns it backwards, with some surprising results.

My past research has focused on three areas. My dissertation [12] focused on the aforementioned inverse limits of elementary embeddings between models of set theory and also on the large cardinal hierarchy near a high-jump cardinal. Previous to my dissertation, I collaborated with Joel Hamkins and Greg Kirmayer to research generalizations of the Kunen inconsistency [7], knocking a few stories off the top of the tower of large cardinal axioms. I am currently working on revising and extending my dissertation research and publishing it as journal articles [13] [14], while at the same time branching out into new research projects in collaboration with my new colleagues at Florida Atlantic University. One of these new projects is to look at elementary epimorphisms between weak models of set theory in collaboration with Robert Lubarsky [11]. Another new project is to learn about algorithmic randomness, so as to be able to collaborate on a project with Katie Brodhead.

2. Brief background on set theory

The large cardinals form a hierarchy of axioms that can be used as a yardstick for measuring the consistency strength of various mathematical propositions. Forcing is an important set-theoretic technique that I use in my work and which is commonly studied in conjunction with large cardinals. The technique of forcing was invented by Paul Cohen to prove the independence of the continuum hypothesis from ZFC. Forcing is somewhat analogous to field extension – it involves extending a ground model of ZFC, often denoted by M, to a forcing extension, $M[G]$, by constructing the minimal model of ZFC that contains both the ground model M and the generic filter G.

3. Inverse limits of elementary embeddings of models of set theory

Although direct limits in the category of elementary embeddings and models of set theory are pervasive in the set-theoretic literature, the inverse limits in this same category have seen less attention, with some notable exceptions such as Richard Laver's study of rank-into-rank embeddings [10]. Since these inverse limits have been comparatively neglected, low-hanging fruit still remains in their study. Furthermore, since the direct limits in this category are quite popular, it is likely that applications can be found for these inverse limits.

I was inspired to study inverse limits when a backwards chain of elementary embeddings of the form in the diagram below arose in my joint work with Joel David Hamkins and Greg Kirmayer in the proof of Theorem 8 of Section 5. For the rest of this section, when the context is clear, I will drop the adjective *backwards* and simply refer to such a structure as a *chain.*

$$\cdots \xrightarrow{j_2} M_2 \xrightarrow{j_1} M_1 \xrightarrow{j_0} M_0$$

Research Goal. *Fully characterize the existence and structure of inverse limits of chains of elementary embeddings of models of set theory of the form above. More generally, extend these results to inverse-directed systems.*

The theory of inverse limits of elementary embeddings of models of set theory is more complicated than that of the direct limits in the same category. On the one hand, every forwards chain in this category has a direct limit, and the domain of this direct limit is always given by the collection of coherent sequences through the chain, which I call the *thread class.* On the other hand, not every backwards chain has an inverse limit, as shown by Theorem 1 below. (Of course, every backwards chain has a thread class, but this thread class may not map *elementarily* into the chain.) The proof of Theorem 1 involves Prikry forcing.

Theorem 1 (Perlmutter [12, Theorem 28], [13])**.** *If there exists a measurable cardinal, then there exists a chain of elementary embeddings of models of* ZFC *with no inverse limit.*

Even when an inverse limit exists, its domain need not be the full thread class; this domain could instead be a proper subclass of the thread class. In particular, in collaboration with Menachem Magidor, I have proven the following.

Theorem 2 (Perlmutter [12, Theorem 29], [13])**.** *Relative to the existence of a* 1*-extendible cardinal, it is consistent that there exists a chain of elementary embeddings of models of* ZFC *for which the inverse limit exists and is a proper subset of the thread class.*

The proof of Theorem 2 uses the technique of iterated Prikry forcing and makes use of the following lemma.

Lemma 3 (Perlmutter [12, Lemma 31], [13])**.** *Suppose κ is a measurable cardinal, G is a generic filter for Prikry forcing with respect to κ, and there is an inaccessible cardinal θ such that $V_\kappa \prec V[G]_\theta$. Then in $V[G]$, there exists a chain of models of* ZFC *and elementary embeddings which has an inverse limit not equal to its thread class.*

I have also constructed several examples of chains that *do* have inverse limits. A sweeping class of examples is as follows – any chain in which every model satisfies $V = \mathrm{HOD}$ must have an inverse limit equal to its thread class. In particular, the chains constructed by the embeddings arising from $0^\#$ satisfy this property.

One particularly illuminating example ties together direct and inverse limits through the use of iterated ultrapowers. Suppose κ is a measurable cardinal with normal measure U. Let M_ω be the ωth model produced from the system of iterated ultrapowers generated by U. Then $j := j_U \upharpoonright M_\omega$ maps the model M_ω elementarily into itself, and the chain in the diagram below has an inverse limit with domain equal to its thread class.

$$\cdots \xrightarrow{\;j\;} M_\omega \xrightarrow{\;j\;} M_\omega \xrightarrow{\;j\;} M_\omega$$

In examples like this, where the models and embeddings are all the same, the thread class is the same as the class of fixed points of the embedding.

While the previous example requires the hypothesis of a measurable cardinal, the existence of a chain of elementary embeddings with an inverse limit actually has no large cardinal strength. Indeed, I have also constructed an example of a chain of (ill-founded) ZFC models with an inverse limit using only the axioms ZFC + Con(ZFC). (This is the minimal consistency strength needed for any model of ZFC to exist.) This example, constructed in collaboration with Victoria Gitman, makes use of ideas about computably saturated models of set theory found in [6].

However, the following question is still open.

Question 4. *What is the large cardinal strength, if any, of the existence of a chain of elementary embeddings of models of* ZFC *with no inverse limit?*

If ZFC is replaced with ZF in Question 4, then the nonexistence of an inverse limit has no large cardinal strength; a chain with no inverse limit can be built using a construction due to Caicedo involving $L(\mathbb{R})$ and Cohen forcing (see [5] and [4]). However, for ZFC models, my proof of Theorem 1 uses a measurable cardinal, and I have no example of a chain of ZFC models without an inverse limit under any weaker hypothesis.

One would expect that a standard argument analogous to that of the Levy-Solovay theorem would show that inverse limits are preserved by forcing in both directions. This is roughly true, although some technicalities are involved. An informal statement of this forcing preservation result is as follows.

Theorem 5 (Perlmutter [12, Theorems 36 and 37], [13])**.** *Suppose that the global lifting criterion holds, giving rise to the ground system of M_is and the lifted system of $M_i[G_i]$s in the diagram below. Then the lifted system has an inverse limit, $N[G]$, if and only if the ground system has an inverse limit, N. In this case, the lift of the inverse limit is the inverse limit of the lifted system, and the ground of the inverse limit is the inverse limit of the grounds.*

$$\begin{array}{ccccccccc}
N[G] & \dashrightarrow & \cdots & \xrightarrow{\;j_2\;} & M_2[G_2] & \xrightarrow{\;j_1\;} & M_1[G_1] & \xrightarrow{\;j_0\;} & M_0[G_0] \\
\cup & & & & \cup & & \cup & & \cup \\
N & \dashrightarrow & \cdots & \xrightarrow{\;j_2\;} & M_2 & \xrightarrow{\;j_1\;} & M_1 & \xrightarrow{\;j_0\;} & M_0
\end{array}$$

4. Large cardinals between supercompact and almost-huge

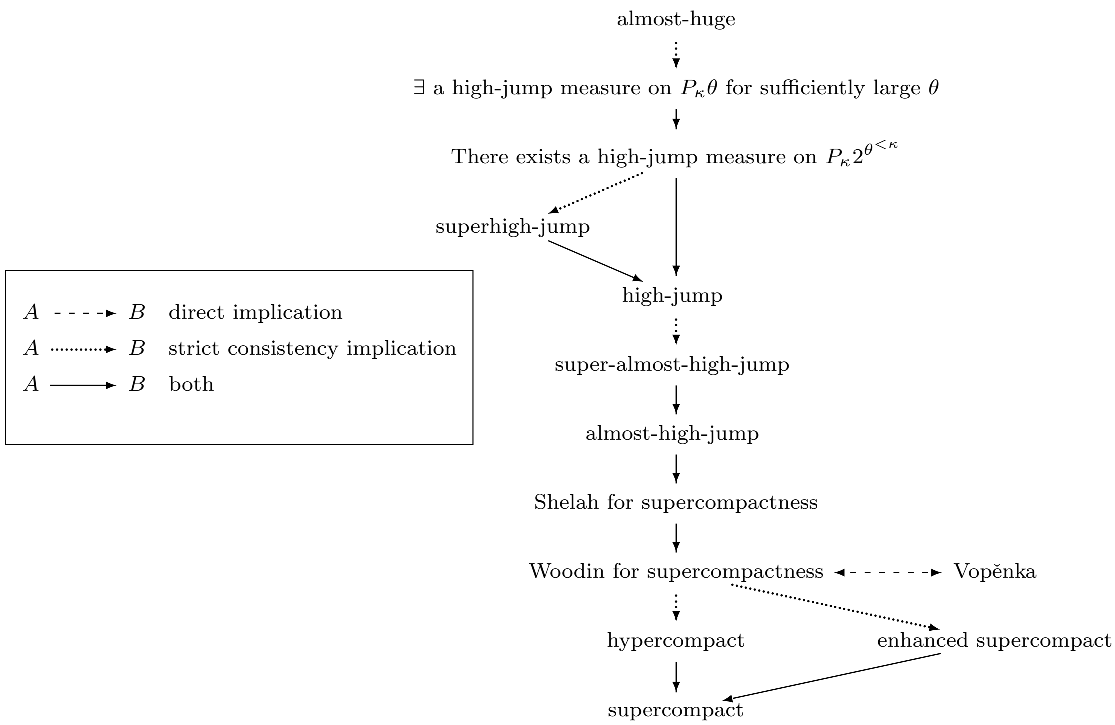
In [3], Arthur Apter and Joel David Hamkins produced a model of set theory with a supercompact cardinal in which every supercompact and partially supercompact cardinal γ is indestructible by $<\gamma$-directed closed forcing. To prove this result, they started with the hypothesis of a high-jump cardinal, defined as follows.

Definition 6. The cardinal κ is **high-jump** if and only if there exists an elementary embedding $j : V \to M$ such that $M^\theta \subseteq M$, where $\theta = \sup\{\, j(f)(\kappa) \mid f : \kappa \to \kappa \,\}$.

This was the first major application of high-jump cardinals, which were introduced as A_4 cardinals in [16, p. 111]. Later refinements of this indestructibility result involved weaker large cardinals, essentially all of which fall between a supercompact cardinal and an almost-huge cardinal.

Research Goal. *Organize the large cardinals between a supercompact cardinal and an almost-huge cardinal by consistency and implicational strength. Generate new large cardinal definitions in this range by weakening or modifying existing definitions. Prove the standard sorts of results relating these cardinals to forcing.*

In [14] and [12], I make substantial progress towards this goal. In particular, the chart below summarizes the relationships among many of the large cardinals between a supercompact cardinal and almost-huge cardinal.



One particularly noteworthy result from the chart is as follows. This result was hinted at by Kanamori in [8, p. 364], but to the best of my knowledge, I am the first to formally prove it.

Theorem 7 (Perlmutter [14, Theorem 5.10], [12, Theorem 94])**.** *A cardinal κ is Vopěnka if and only if κ is Woodin for supercompactness.*

On the chart, there are some large cardinal relationships that have not been fully discovered. In particular, I would like to do more work with two large cardinals invented by Apter: hypercompact cardinals and enhanced supercompact cardinals (See [2] and [1]). I discovered a subtle error in Apter's original published definition of a hypercompact cardinal, and this definition turned out to be inconsistent. However, for Apter's applications of a hypercompact cardinal, a weaker version of hypercompactness suffices. There is still more work to be done in figuring out the relationship between hypercompact cardinals and enhanced supercompact cardinals. I would also like to determine how both of these relate to extendible cardinals, and I am making progress towards this goal in collaboration with Robert Lubarsky.

Additionally, I have proven several results relating high-jump cardinals and forcing, including analogues of the Levy-Solovay theorem and the Laver diamond principle for high-jump cardinals.

5. Generalizations of the Kunen Inconsistency

In 1971, Kenneth Kunen famously proved that, assuming AC, a nontrivial elementary embedding $j : V \to V$ cannot exist [9]. This result is known as the Kunen inconsistency and puts an upper bound on the large cardinal hierarchy.

In my joint work with Hamkins and Kirmayer, we proved many generalizations of the Kunen inconsistency [7]. These generalizations can be viewed as refuting the existence of certain very large cardinals. One of the most striking of our many generalizations of the Kunen inconsistency is the following, which rules out an extremely broad range of elementary embeddings.

Theorem 8 ([7, Theorem 28])**.** *Let M be an inner model of* ZFC *definable in V without parameters. Then there does not exist a nontrivial elementary embedding $j : M \to V$.*

Many interesting questions remain in this area of study, including the question as to whether a nontrivial elementary embedding $j : V \to V$ can exist in the absence of AC.

6. Elementary epimorphisms

I recently began studying elementary epimorphisms between models of set theory in collaboration with Lubarsky [11]. Elementary epimorphisms were introduced by Philipp Rothmaler in [15]. An elementary epimorphism $f : M \to N$ between two models with the same signature is a surjective homomorphism with the additional property that it is a backwards version of an elementary embedding: whenever $N \models \varphi(y_1, \dots y_n)$, there exist $x_1, \dots x_n \in M$ such that $f(x_i) = y_i$ and such that $M \models \varphi(x_1, \dots x_n)$. Rothmaler studied inverse limits of elementary epimorphisms, primarily in the context of modules over a ring. Accordingly, upon hearing about my study of inverse limits of elementary embeddings of models of set theory, he encouraged me to consider inverse limits of elementary epimorphisms between models of set theory. Lubarsky and I then proved the following theorem.

Theorem 9. *Every elementary epimorphism between models of* ZF *is an isomorphism. In particular, every elementary epimorphism between transitive models of* ZF *is the identity function.*

However, more interesting elementary epimorphisms may exist between weaker models of set theory. Along these lines, Lubarsky and I are working to prove the following conjecture.

Conjecture 10. *It is consistent relative to* ZFC *that there exists a nonisomorphic elementary epimorphism between two transitive models of Kripke-Platek set theory.*

If we are successful in proving this conjecture, then we may next consider inverse limits of such embeddings. Additionally, I would like to get a grip on which particular axioms of set theory preclude the existence of nonisomorphic elementary epimorphisms. This research on elementary epimorphisms builds on a theme of my research on inverse limits, turning concepts related to elementary embeddings backwards. It also builds on a theme of my research in generalizations of the Kunen inconsistency, proving that certain structure-preserving functions between particular classes of models of set theory must be trivial.

References

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