A Journey Through the World of Mice and Games

Projective and Beyond

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Young Set Theory Workshop Copenhagen

- Descriptive Set Theory
- Inner Model Theory
- Beyond the Projective Hierarchy

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Games in Set Theory

Definition (Gale/Stewart 1953)

Let $A \subset 2^{\mathbb{N}}$. With $G(A)$ we denote the following game

$$\begin{array}{c|cccccc} \mathrm{I} & i_0 & & i_2 & & \dots & \\ \hline \mathrm{II} & & i_1 & & i_3 & & \dots \end{array}$$

for $i_n \in \{0,1\}$ and $n \in \mathbb{N}$.

We say player I wins the game iff $(i_n)_{n\in\mathbb{N}} \in A$. Otherwise player II wins. We say $G(A)$ (or A itself) is *determined* iff one of the players has a winning strategy (in the obvious sense).

Which games are determined?

Theorem (Gale/Stewart, 1953)

(AC) Let $A \subset 2^{\mathbb{N}}$ be open or closed. Then $G(A)$ is determined.

Theorem (Gale/Stewart, 1953)

Assuming AC there is a set of reals which is not determined.

The Projective Hierarchy

Let $\mathbb{B}$ be the collection of all Borel sets of reals. Then we define the projective hierarchy as follows.

$$\begin{aligned} \mathbf{\Sigma}^1_1 = & \text{ analytic sets, i.e. projections of Borel sets,} \\ \mathbf{\Pi}^1_n = & \text{ complements of sets in } \mathbf{\Sigma}^1_n, \\ \mathbf{\Sigma}^1_{n+1} = & \text{ projections of sets in } \mathbf{\Pi}^1_n. \end{aligned}$$

A set is projective if it is in $\mathbf{\Sigma}^1_n$ (or $\mathbf{\Pi}^1_n$) for some n.

Determinacy for Different Sets of Reals

Theorem (Martin, 1975)

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Determinacy for all projective sets of reals is not provable in ZFC alone.

Theorem (Martin/Steel, 1985)

Assume ZFC and there are n Woodin cardinals with a measurable cardinal above them all. Then every $\mathbf{\Sigma}^1_{n+1}$ set is determined.

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Inner Model Theory

The main goal of inner model theory is to construct L-like models, which we call mice, for stronger and stronger large cardinals.

Gödel's constructible universe L

Definition

Let E be a set or a proper class. Let

$$\begin{aligned} J_0[E] &= \emptyset \\ J_{\alpha+1}[E] &= \mathrm{rud}_E(J_\alpha[E] \cup \{J_\alpha[E]\}) \\ J_\lambda[E] &= \bigcup_{\alpha<\lambda} J_\alpha[E] \text{ for limit } \lambda \\ L[E] &= \bigcup_{\alpha\in\mathrm{Ord}} J_\alpha[E] \end{aligned}$$

Note that rud_E denotes the closure under functions which are rudimentary in E (i.e. basic set operations like minus, union and pairing or intersection with E).

Basic properties of L

Condensation Let α be an infinite ordinal and let

$$M \prec (L_\alpha, \in).$$

Then the transitive collapse of M is equal to L_β for some ordinal $\beta \leq \alpha$.

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Comparison Let L_α and L_β for ordinals α and β be initial segments of L. Then one is an initial segment of the other, that means

$$L_\alpha \trianglelefteq L_\beta \text{ or } L_\beta \trianglelefteq L_\alpha.$$

Basic Concepts of Inner Model Theory

Definition

Let $\mathcal{M}$ be a countable model of set theory, κ a cardinal and $\mathcal{U}$ a κ-complete, nonprincipal ultrafilter on $\mathcal{M}$. Then there is a transitive model $\mathcal{N} = \mathrm{Ult}(\mathcal{M}, \mathcal{U})$ and an elementary embedding $i_{\mathcal{U}} : \mathcal{M} \to \mathcal{N}$ with critical point κ. We call $\mathcal{N}$ the *ultrapower* of $\mathcal{M}$ via $\mathcal{U}$.

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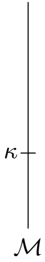

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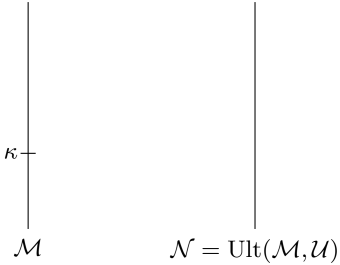

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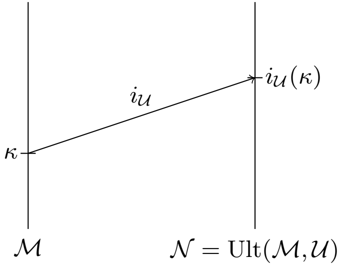

Mitchell and Jensen generalized the concept of measures to extenders to obtain stronger ultrapowers.

Definition

Let $\mathcal{M}$ be a countable model of set theory. An *extender* over $\mathcal{M}$ is a system of ultrafilters whose ultrapowers form a directed system, such that they give rise to a single elementary embedding.

In fact for every embedding $j : \mathcal{M} \to \mathcal{N}$ there is an extender E over $\mathcal{M}$ which gives rise to this embedding.

Comparison

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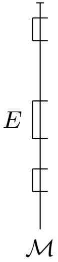

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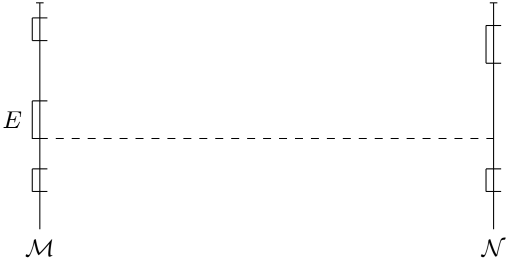

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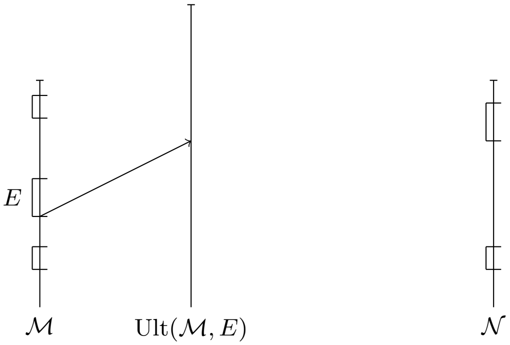

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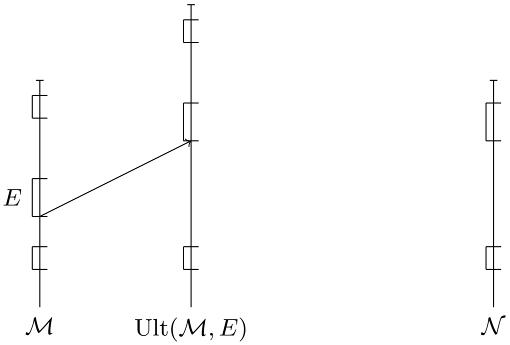

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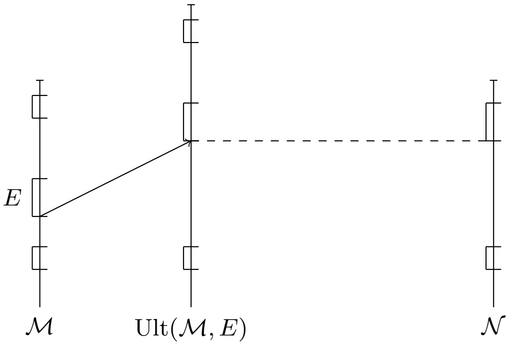

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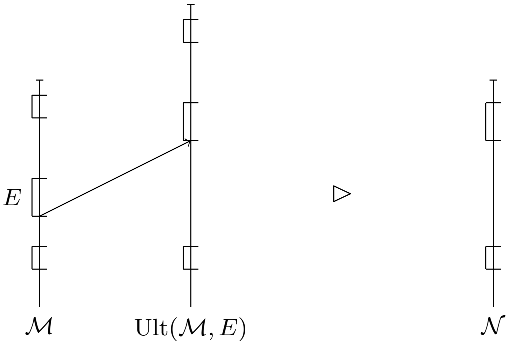

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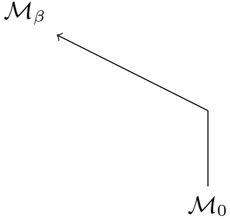

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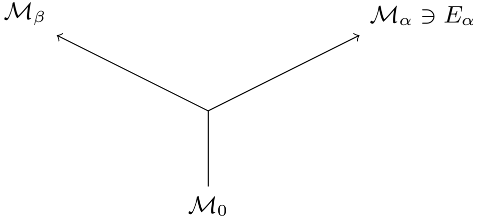

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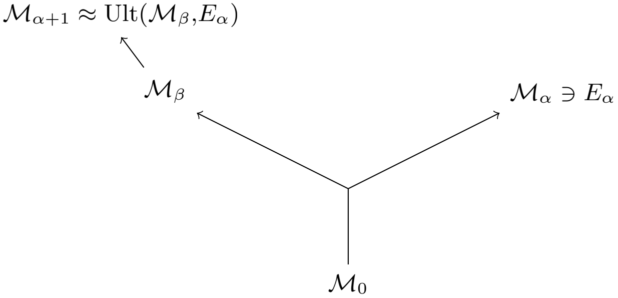

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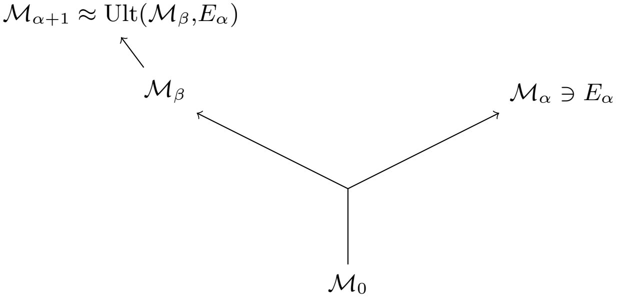

The central problem is to choose a cofinal branch such that the direct limit is well-founded.

The iteration game

More precisely we consider the following two player game $\mathcal{G}(\mathcal{M}, \omega_1)$ of length $< \omega_1$ for a premouse $\mathcal{M}$.

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Definition

We say a premouse $\mathcal{M}$ is ω_1-iterable iff player II has a winning strategy in the game $\mathcal{G}(\mathcal{M}, \omega_1)$. This winning strategy is called an iteration strategy for $\mathcal{M}$.

Iterable Models and Projective Determinacy

Theorem (Neeman, Woodin)

Let $n \geq 1$. Then the following are equivalent.

(a) $\mathbf{\Sigma}^1_{n+1}$*-determinacy.*

(b) *For every $x \in \mathbb{R}$ the ω_1-iterable countable model of set theory with n Woodin cardinals $M_n^\#(x)$ exists.*

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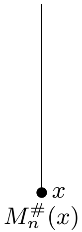

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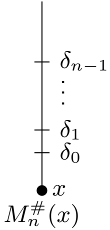

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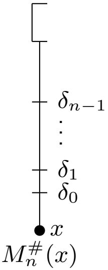

- Descriptive Set Theory
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The $L(\mathbb{R})$-hierarchy

We define a hierarchy in $L(\mathbb{R})$ as follows.

$$\begin{aligned}
J_1(\mathbb{R}) &= V_{\omega+1} \\
J_{\alpha+1}(\mathbb{R}) &= \mathrm{rud}(J_\alpha(\mathbb{R}) \cup \{J_\alpha(\mathbb{R})\}) \\
J_\lambda(\mathbb{R}) &= \bigcup_{\alpha<\lambda} J_\alpha(\mathbb{R}) \text{ for limit } \lambda \\
L(\mathbb{R}) &= \bigcup_{\alpha\in\mathrm{Ord}} J_\alpha(\mathbb{R})
\end{aligned}$$

The $L(\mathbb{R})$-hierarchy extends the projective hierarchy

We can consider a hierarchy of pointclasses like

$$\boldsymbol{\Sigma}_n^{J_\beta(\mathbb{R})}$$

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Recall that $J_1(\mathbb{R}) = V_{\omega+1}$. We in fact have that

$$\mathbf{\Sigma}_n^{J_1(\mathbb{R})} \cap \mathcal{P}(\mathbb{R}) = \mathbf{\Sigma}_n^1,$$

so the $L(\mathbb{R})$-hierarchy extends the projective hierarchy.

Pointclasses beyond the projective hierarchy

In some sense this hierarchy looks like infinitely many copies of the projective hierarchy, but between these copies we might have different forms of *gaps*.

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Definition

We say $[\alpha, \beta]$ is a Σ_1-gap iff

(i) $J_\alpha(\mathbb{R}) \prec_{\Sigma_1} J_\beta(\mathbb{R})$,

(ii) for all $\alpha' < \alpha$, $J_{\alpha'}(\mathbb{R}) \not\prec_{\Sigma_1} J_\alpha(\mathbb{R})$, and

(iii) for all $\beta' > \beta$, $J_\beta(\mathbb{R}) \not\prec_{\Sigma_1} J_{\beta'}(\mathbb{R})$.

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- So maybe for the $L(\mathbb{R})$-hierarchy we have to consider mice with infinitely many Woodin cardinals.
- No!

Theorem (Woodin)

The following are equiconsistent.

(1) *There exist infinitely many Woodin cardinals.*

(2) $L(\mathbb{R})$*-determinacy.*

Mice beyond the projective hierarchy

The mice we want to construct from determinacy at these levels also have finitely many Woodin cardinals. What gives them strength here is that we want them to *capture* certain sets of reals.

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$$A \cap \mathcal{P}[g] = i(\tau)^g.$$

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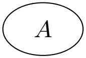

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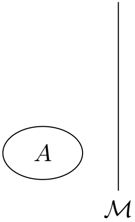

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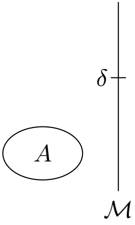

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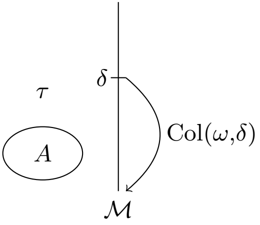

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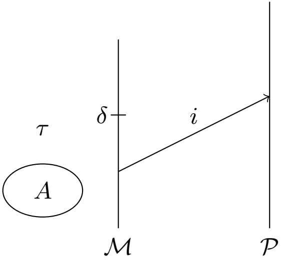

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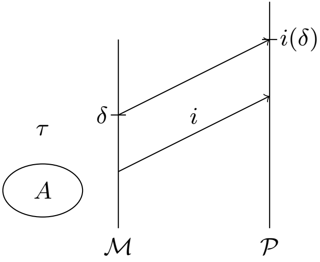

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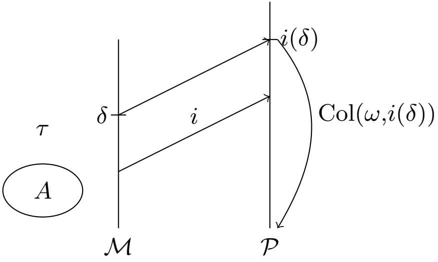

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Let $A \subset \mathbb{R}$, let $\mathcal{M}$ be a countable mouse and δ an uncountable cardinal of $\mathcal{M}$. Then we say $\mathcal{M}$ *captures* A *at* δ iff there is an $\mathcal{M}$-term τ for A at δ and $\mathcal{M}$ absorbs reals at δ.

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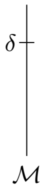

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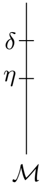

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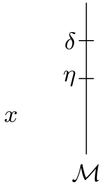

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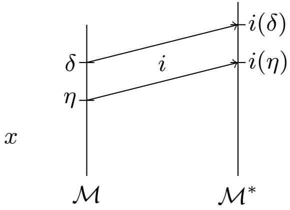

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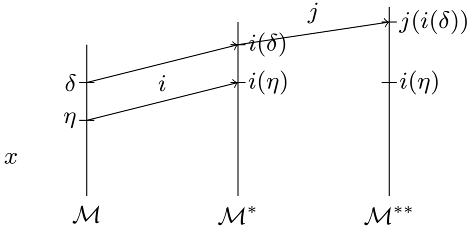

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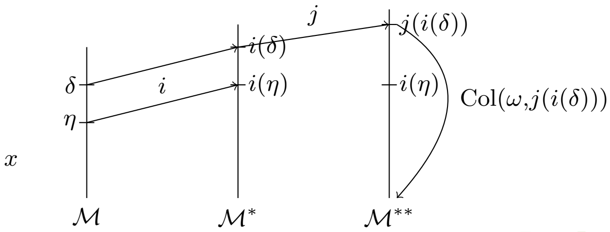

Examples of mice capturing sets of reals

Lemma

The ω_1-iterable countable model of set theory with n Woodin cardinals $M_n^{\#}(x)$ captures $\boldsymbol{\Sigma}^1_{n+1}$-sets of reals at its bottom Woodin cardinal.

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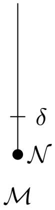

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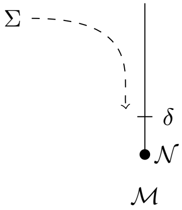

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Note that we need an iteration strategy Σ that behaves nicely and we have to be careful during the construction to make sure that we obtain a reasonable model.

Hybrid mice from determinacy

Theorem

Let $\alpha < \beta$ be ordinals such that $[\alpha, \beta]$ is a weak Σ_1-gap, let $k \geq 0$, and let

$$A \in \Gamma = \mathbf{\Sigma}_n(J_\beta(\mathbb{R})) \cap \mathcal{P}(\mathbb{R}),$$

where $n < \omega$ is the least natural number such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$. Moreover assume that every $\mathbf{\Pi}^1_{2k+5}\Gamma$-definable set of reals is determined. Then there exists an ω_1-iterable hybrid Σ-premouse $\mathcal{N}$ which captures every set of reals in the pointclass $\Sigma^1_k(A)$ or $\Pi^1_k(A)$.

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The mouse $\mathcal{M}$ we construct is in fact of the form $M_k^{\Sigma,\#}(\mathcal{N})$ for some countable mouse $\mathcal{N}$.

Open questions

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Conjecture (Steel, Woodin, 2015)

Let $\mathcal{P}$ be the minimal ladder mouse. Then for all reals x, we have that

$$x \in \mathcal{P} \cap \mathbb{R} \text{ iff } x \text{ is } \boldsymbol{\Delta}_2^{J_2(\mathbb{R})}\text{-definable in a countable ordinal.}$$

Thank you for your attention!

For reference see "Pure and Hybrid Mice with Finitely Many Woodin Cardinals from Levels of Determinacy" (Dissertation), soon available at http://boolesrings.org/sandrauhlenbrock/publications/dissertation/