

MICE WITH FINITELY MANY WOODIN CARDINALS FROM OPTIMAL DETERMINACY HYPOTHESES

RALF SCHINDLER, SANDRA UHLENBROCK, AND W. HUGH WOODIN

ABSTRACT

We prove the following result which is due to the third author.
 Let $n \geq 1$. If $\mathbf{\Pi}_n^1$ determinacy and $\mathbf{\Pi}_{n+1}^1$ determinacy both hold true and there is no Σ_{n+2}^1 -definable ω_1 -sequence of pairwise distinct reals, then $M_n^\#$ exists and is ω_1 -iterable. The proof yields that $\mathbf{\Pi}_{n+1}^1$ determinacy implies that $M_n^\#(x)$ exists and is ω_1 -iterable for all reals x .
 A consequence is the Determinacy Transfer Theorem for arbitrary $n \geq 1$, namely the statement that $\mathbf{\Pi}_{n+1}^1$ determinacy implies $\mathfrak{D}^{(n)}(< \omega^2 - \mathbf{\Pi}_1^1)$ determinacy.

CONTENTS

Abstract	1
Preface	2
Overview	4
1. Introduction	6
1.1. Games and Determinacy	6
1.2. Inner Model Theory	8
1.3. Mice with Finitely Many Woodin Cardinals	9
1.4. Mice and Determinacy	15
2. A Model with Woodin Cardinals from Determinacy Hypotheses	16
2.1. Introduction	16
2.2. Preliminaries	17
2.3. OD-Determinacy for an Initial Segment of M_{n-1}	33
2.4. Applications	41
2.5. A Proper Class Inner Model with n Woodin Cardinals	46
3. Proving Iterability	51
3.1. Existence of n -suitable Premice	51

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3.2. Correctness for n -suitable Premice	64
3.3. Outline of the Proof	69
3.4. Making use of the Right Hypothesis	70
3.5. $M_{2n-1}^\#(x)$ from Boldface Π_{2n}^1 Determinacy	74
3.6. $M_{2n}^\#(x)$ from Boldface Π_{2n+1}^1 Determinacy	87
4. Conclusion	119
4.1. Applications	120
4.2. Open problems	121
References	121

PREFACE

The research area of Set Theory goes back to Georg Cantor, who phrased many essential questions and provided a framework for further research in a series of papers in the late 19th century. During the beginning of the 20th century Ernst Zermelo and Abraham Fraenkel developed an axiomatic system for Set Theory called ZF, that is still fundamental today. Together with the Axiom of Choice it constitutes ZFC, the central background theory for modern Mathematics and Set Theory.

Already in 1878 Georg Cantor asked the very natural question of whether there is a set of size strictly between that of the natural numbers and that of the continuum. This question, known as the *Continuum Problem*, motivated lots of research in Set Theory since then and continues to do so, for example in cardinal arithmetic. It was proved to be independent of ZFC by work of Kurt Gödel on his model L in 1940 and Paul Cohen's development of *forcing* in 1963 and hence is an example which shows that the axiomatic system ZFC is not sufficient to solve all interesting questions about sets.

Therefore one of the main goals of research in Set Theory can be phrased as the *search for the "right" axioms for mathematics*. This means in particular studying various extensions of ZFC and their properties. The work in this paper is related to this overall goal, because we study different extensions of ZFC and their relationships.

In 1953 David Gale and Frank M. Stewart developed a basic theory of infinite games in [GS53]. For every set of reals, that means set of sequences of natural numbers, A they considered a two-player game $G(A)$ of length ω , where player I and player II alternately play natural numbers. They defined that player I wins the game $G(A)$ if and only if the sequence of natural numbers produced during a run of the game $G(A)$ is contained in A and otherwise player II wins. Moreover the game $G(A)$ or the set A itself is called determined if and only if one of the two players has a winning strategy, that

means a method by which they can win, no matter what their opponent does, in the game described above.

Already in [GS53] the authors were able to prove that every open and every closed set of reals is determined using ZFC. But furthermore they proved that determinacy for all sets of reals contradicts the Axiom of Choice. This leads to the natural question of what the picture looks like for definable sets of reals which are more complicated than open and closed sets. After some partial results by Philip Wolfe in [Wo55] and Morton Davis in [Da64], Donald A. Martin was finally able to prove in [Ma75] from ZFC that every Borel set of reals is determined.

In the meantime the development of so called *Large Cardinal Axioms* was proceeding in Set Theory. In 1930 Stanisław Ulam first defined measurable cardinals and at the beginning of the 1960's H. Jerome Keisler [Kei62] and Dana S. Scott [Sc61] found a way of making them more useful in Set Theory by reformulating the statements using elementary embeddings.

About the same time, other set theorists were able to prove that *Determinacy Axioms* imply *regularity properties* for sets of reals. More precisely Banach and Mazur showed that under the Axiom of Determinacy (that means every set of reals is determined), every set of reals has the Baire property. Mycielski and Swierczkowski proved in [MySw64] that under the same hypothesis every set of reals is Lebesgue measurable. Furthermore Davis showed in [Da64] that under this hypothesis every set of reals has the perfect set property. Moreover all three results also hold if the Determinacy Axioms and regularity properties are only considered for sets of reals in certain pointclasses. This shows that Determinacy Axioms have a large influence on the structure of sets of reals and therefore have a lot of applications in Set Theory.

In 1965 Robert M. Solovay was able to prove these regularity properties for a specific pointclass, namely Σ_2^1 , assuming the existence of a measurable cardinal instead of a Determinacy Axiom (see for example [So69] for the perfect set property). Then finally Donald A. Martin was able to prove a direct connection between Large Cardinals and Determinacy Axioms: he showed in 1970 that the existence of a measurable cardinal implies determinacy for every analytic set of reals (see [Ma70]).

Eight years later Leo A. Harrington established that this result is in some sense optimal. In [Ha78] he proved that determinacy for all analytic sets implies that $0^\#$, a countable active mouse which can be obtained from a measurable cardinal, exists. Here a mouse is a fine structural, iterable model. Together with Martin's result mentioned above this yields that the two statements are in fact equivalent.

This of course motivates the question of whether a similar result can be obtained for larger sets of reals, so especially for determinacy in the projective hierarchy. The right large cardinal notion to consider for these sets of reals was introduced by the third author in 1984 and is nowadays called a Woodin

cardinal. Building on this, Donald A. Martin and John R. Steel were able to prove in [MaSt89] almost twenty years after Martin’s result about analytic determinacy that, assuming the existence of n Woodin cardinals and a measurable cardinal above them all, every Σ_{n+1}^1 -definable set of reals is determined.

In the meantime the theory of mice was further developed. At the level of strong cardinals it goes back to Ronald B. Jensen, Robert M. Solovay, Tony Dodd and Ronald B. Jensen, and William J. Mitchell. Then it was further extended to the level of Woodin cardinals by Donald A. Martin and John R. Steel in [MaSt94] and William J. Mitchell and John R. Steel in [MS94], where some errors were later repaired by John R. Steel, Martin Zeman, and the first author in [SchStZe02]. Moreover Ronald B. Jensen developed another approach to the theory of mice at the level of Woodin cardinals in [Je97].

In 1995 Itay Neeman was able to improve the result from [MaSt89] in [Ne95]. He showed that the existence and ω_1 -iterability of $M_n^\#$, the minimal countable active mouse at the level of n Woodin cardinals, is enough to obtain that every $\mathcal{D}^n(< \omega^2 - \Pi_1^1)$ -definable set of reals is determined. Here “ \mathcal{D} ” is a quantifier which is defined by a game. Neeman’s result implies that if for all reals x the premouse $M_n^\#(x)$ exists and is ω_1 -iterable, then in particular every Σ_{n+1}^1 -definable set of reals is determined. For odd n this latter result was previously known by the third author. The converse of this latter result was announced by the third author in the 1980’s but a proof has never been published. The goal of this paper is to finally provide a complete proof of his result.

OVERVIEW

The purpose of this paper is to give a proof of the following theorem, which connects inner models with Woodin cardinals and descriptive set theory at projective levels in a direct level-wise way. This theorem is due to the third author and announced for example in the addendum (§5) of [KW08] and in Theorem 5.3 of [Sch10], but so far a proof of this result has never been published.

Theorem 2.1. *Let $n \geq 1$ and assume Π_{n+1}^1 determinacy holds. Then $M_n^\#(x)$ exists and is ω_1 -iterable for all $x \in {}^\omega\omega$.*

The converse of Theorem 2.1 also holds true and is for odd n due to the third author (unpublished) and for even n due to Itay Neeman (see [Ne95]). From this we can obtain the following corollary, which is Theorem 1.10 in [KW08] for odd n . We will present a proof of the Determinacy Transfer Theorem for the even levels n as a corollary of Theorem 2.1 in Section 4.1, using Theorem 2.5 in [Ne95] due to Itay Neeman.

Corollary 4.1 (Determinacy Transfer Theorem). *Let $n \geq 1$. Then Π_{n+1}^1 determinacy is equivalent to $\mathfrak{D}^{(n)}(< \omega^2 - \Pi_1^1)$ determinacy.*

In fact we are going to prove the following theorem which will imply Theorem 2.1.

Theorem 3.14. *Let $n \geq 1$ and assume there is no Σ_{n+2}^1 -definable ω_1 -sequence of pairwise distinct reals. Moreover assume that Π_n^1 determinacy and Π_{n+1}^1 determinacy hold. Then $M_n^\#$ exists and is ω_1 -iterable.*

This is a part of the following theorem.

Theorem 3.13. *Let $n \geq 1$ and assume there is no Σ_{n+2}^1 -definable ω_1 -sequence of pairwise distinct reals. Then the following are equivalent.*

- (1) Π_n^1 determinacy and Π_{n+1}^1 determinacy,
- (2) for all $x \in {}^\omega\omega$, $M_{n-1}^\#(x)$ exists and is ω_1 -iterable, and $M_n^\#$ exists and is ω_1 -iterable,
- (3) $M_n^\#$ exists and is ω_1 -iterable.

Here the direction (3) implies (1) follows from Theorem 2.14 in [Ne02] and is due to the third author for odd n (unpublished) and due to Itay Neeman for even n . Moreover the direction (2) implies (1) and the equivalence of (2) and (3) as proven in [Ne02] do not need the background hypothesis that there is no Σ_{n+2}^1 -definable ω_1 -sequence of pairwise distinct reals.

Furthermore we get the following relativized version of Theorem 3.13.

Corollary 2.2. *Let $n \geq 1$. Then the following are equivalent.*

- (1) Π_{n+1}^1 determinacy, and
- (2) for all $x \in {}^\omega\omega$, $M_n^\#(x)$ exists and is ω_1 -iterable.

This gives that Π_{n+1}^1 determinacy is an optimal hypothesis for proving the existence and ω_1 -iterability of $M_n^\#(x)$ for all $x \in {}^\omega\omega$.

Remark. In contrast to the statement of Theorem 1.4 in [Ne02] it is open whether Π_n^1 determinacy and Π_{n+1}^1 determinacy alone imply the existence of an ω_1 -iterable $M_n^\#$ for $n > 1$ (see also Section 4.2). Whenever we are citing [Ne02] in this paper we make no use of any consequence of this result stated there.

Outline. This paper is organized as follows. In Section 1 we give a short introduction to determinacy and inner model theory. In particular we state some known results concerning the connection of determinacy for certain sets of reals and the existence of mice with large cardinals.

In Section 2 we will construct a proper class inner model with n Woodin cardinals from Π_n^1 determinacy and Π_{n+1}^1 determinacy. For that purpose we will prove in Lemma 2.3 from the same determinacy hypothesis that for

a cone of reals x the premouse $M_{n-1}(x)|\delta_x$ is a model of OD-determinacy, where δ_x denotes the least Woodin cardinal in $M_{n-1}(x)$. This generalizes a theorem of Kechris and Solovay to the context of mice (see Theorem 3.1 in [KS85]).

Afterwards we will prove in Section 3 that, assuming Π_{n+1}^1 determinacy, there is in fact an ω_1 -iterable model which has n Woodin cardinals. More precisely, we will prove under this hypothesis that $M_n^\#(x)$ exists and is ω_1 -iterable for all reals x . The proof of this result divides into different steps. In Sections 3.1 and 3.2 we will introduce the concept of n -suitable premice and show that if n is odd, using the results in Section 2, such n -suitable premice exist assuming Π_n^1 determinacy and Π_{n+1}^1 determinacy. The rest of Section 3 will also be divided into different cases depending if n is even or odd.

In Section 3.4 we will show that Π_{n+1}^1 determinacy already implies that every Σ_{n+2}^1 -definable sequence of pairwise distinct reals is countable. The proof of Theorem 2.1 is organized as a simultaneous induction for odd and even levels n . We will show in Section 3.5 that the hypothesis that there is no Σ_{2n+1}^1 -definable ω_1 -sequence of pairwise distinct reals in addition to Π_{2n-1}^1 determinacy and Π_{2n}^1 determinacy suffices to prove that $M_{2n-1}^\#$ exists and is ω_1 -iterable. This finishes the proof of Theorem 2.1 for the odd levels n of the inductive argument. In Section 3.6 we will finally prove the analogous result for even n , that means we will show that if every Σ_{2n+2}^1 -definable sequence of pairwise distinct reals is countable and Π_{2n}^1 determinacy and Π_{2n+1}^1 determinacy hold, then $M_{2n}^\#$ exists and is ω_1 -iterable. The proof is different for the odd and even levels of the projective hierarchy because of the periodicity in terms of uniformization and correctness.

We close this paper with proving the Determinacy Transfer Theorem for even n as an application of Theorem 2.1 and mentioning related open questions in Section 4.

1. INTRODUCTION

In this section we will introduce some relevant notions such as games and mice and their basic properties. In particular we will summarize some known results about the connection between large cardinals and the determinacy of certain games. Then we will have a closer look at mice with finitely many Woodin cardinals and introduce the premouse $M_n^\#$.

1.1. Games and Determinacy. Throughout this paper we will consider games in the sense of [GS53] if not specified otherwise. We will always identify ${}^\omega 2$, ${}^\omega \omega$, and \mathbb{R} with each other, so that we can define Gale-Stewart games as follows.

Definition 1.1 (Gale, Stewart). *Let $A \subseteq \mathbb{R}$. By $G(A)$ we denote the following game.*

$$\begin{array}{c|cccc} \text{I} & i_0 & i_2 & \cdots & \\ \hline \text{II} & & i_1 & i_3 & \cdots \end{array} \quad \text{for } i_n \in \{0, 1\} \text{ and } n \in \omega.$$

We say player I wins the game $G(A)$ iff $(i_n)_{n < \omega} \in A$. Otherwise we say player II wins.

Definition 1.2. Let $A \subseteq \mathbb{R}$. We say $G(A)$ (or the set A itself) is determined iff one of the players has a winning strategy in the game $G(A)$ (in the obvious sense).

Some famous results concerning the question which sets of reals are determined are the following. The first three theorems can be proven in ZFC.

Theorem 1.3 (Gale, Stewart in [GS53]). Let $A \subset \mathbb{R}$ be open or closed and assume the Axiom of Choice. Then $G(A)$ is determined.

Theorem 1.4 (Gale, Stewart in [GS53]). Assuming the Axiom of Choice there is a set of reals which is not determined.

Theorem 1.5 (Martin in [Ma75]). Let $A \subset \mathbb{R}$ be a Borel set and assume the Axiom of Choice. Then $G(A)$ is determined.

To prove stronger forms of determinacy we need to assume for example large cardinal axioms.

Theorem 1.6 (Martin in [Ma70]). Assume ZFC and that there is a measurable cardinal. Let $A \subseteq \mathbb{R}$ be an analytic, i.e. Σ_1^1 -definable, set. Then $G(A)$ is determined.

Determinacy in the projective hierarchy can be obtained from finitely many Woodin cardinals which were introduced by the third author in 1984 and are defined as follows.

Definition 1.7. Let $\kappa < \delta$ be cardinals and $A \subseteq \delta$. Then κ is called A -reflecting in δ iff for every $\lambda < \delta$ there exists a transitive model of set theory M and an elementary embedding $\pi : V \rightarrow M$ with critical point κ , such that $\pi(\kappa) > \lambda$ and

$$\pi(A) \cap \lambda = A \cap \lambda.$$

Definition 1.8. A cardinal δ is called a Woodin cardinal iff for all $A \subseteq \delta$ there is a cardinal $\kappa < \delta$ which is A -reflecting in δ .

Theorem 1.9 (Martin, Steel in [MaSt89]). Let $n \geq 1$. Assume ZFC and that there are n Woodin cardinals with a measurable cardinal above them all. Then every Σ_{n+1}^1 -definable set of reals is determined.

See for example Chapter 13 in [Sch14] or Section 5 in [Ne10] for modern write-ups of the proof of Theorem 1.9.

The existence of infinitely many Woodin cardinals with a measurable cardinal above them all yields a much stronger form of determinacy as shown in the following theorem due to the third author.

Theorem 1.10 ([KW10]). *Assume ZFC and that there are ω Woodin cardinals with a measurable cardinal above them all. Then every set of reals in $L(\mathbb{R})$ is determined.*

The goal of this paper is to prove results in the converse direction. That means we want to obtain large cardinal strength from determinacy axioms. This is done using inner model theoretic concepts which we start to introduce in the next section.

1.2. Inner Model Theory. The most important concept in inner model theory is a mouse. Therefore we briefly review the definition of mice in this section and mention some relevant properties without proving them. The reader who is interested in a more detailed introduction to mice is referred to Section 2 of [St10].

We assume that the reader is familiar with some fine structure theory as expounded for example in [MS94] or [SchZe10].

In general the models we are interested in are of the form $L[\vec{E}]$ for some coherent sequence of extenders \vec{E} . This notion goes back to Ronald B. Jensen and William J. Mitchell and is made more precise in the following definition.

Definition 1.11. *We say M is a potential premouse iff*

$$M = (J_{\eta}^{\vec{E}}, \in, \vec{E} \upharpoonright \eta, E_{\eta})$$

for some fine extender sequence \vec{E} and some ordinal η . We say that such a potential premouse M is active iff $E_{\eta} \neq \emptyset$.

Moreover if $\kappa \leq \eta$, we write

$$M \upharpoonright \kappa = (J_{\kappa}^{\vec{E}}, \in, \vec{E} \upharpoonright \kappa, E_{\kappa}).$$

Here *fine extender sequence* is in the sense of Definition 2.4 in [St10]. This definition of a fine extender sequence goes back to Section 1 in [MS94] and [SchStZe02].

Definition 1.12. *Let M be a potential premouse. Then we say M is a premouse iff every proper initial segment of M is ω -sound.*

We informally say that a *mouse* is an iterable premouse, but since there are several different notions of iterability we try to avoid to use the word “mouse” in formal context, especially if it is not obvious what sort of iterability is meant. Nevertheless whenever it is not specified otherwise “iterable” in this paper always means “ ω_1 -iterable” as defined below and therefore a “mouse” will be an ω_1 -iterable premouse.

Definition 1.13. *We say a premouse M is ω_1 -iterable iff player II has a winning strategy in the iteration game $\mathcal{G}_{\omega}(M, \omega_1)$ as described in Section 3.1 of [St10].*

The iteration trees which are considered in Section 3 in [St10] are called *normal* iteration trees.

Whenever not specified otherwise we will assume throughout this paper that all iteration trees are normal to simplify the notation. Since normal iteration trees do not suffice to prove for example the Dodd-Jensen Lemma (see Section 4.2 in [St10]) it is necessary to consider stacks of normal trees. See Definition 4.4 in [St10] for a formal definition of iterability for stacks of normal trees.

All arguments to follow easily generalize to countable stacks of normal trees of length $< \omega_1$ instead of just normal trees of length $< \omega_1$. The reason for this is that the iterability we will prove in this paper for different kinds of premeice will in fact always be obtained from the sort of iterability for the model K^c which is proven in Chapter 9 in [St96].

Throughout this paper we will use the notation from [St10] for iteration trees.

1.3. Mice with Finitely Many Woodin Cardinals. We first fix some notation and give a short background on the mouse $M_n^\#$. In this paper we always assume $M_n^\#$ to be ω_1 -iterable if not specified otherwise.

The premeice we are going to consider in this paper will mostly have the following form.

Definition 1.14. *Let $n \geq 1$. A premouse M is called n -small iff for every critical point κ of an extender on the M -sequence*

$$M|\kappa \not\models \text{“there are } n \text{ Woodin cardinals”}.$$

Moreover we say that a premouse M is 0-small iff M is an initial segment of Gödel’s constructible universe L .

Moreover ω -small premeice are defined analogously.

Definition 1.15. *Let $n \geq 1$ and $x \in {}^\omega\omega$. Then $M_n^\#(x)$ denotes the unique countable, sound, ω_1 -iterable x -premouse which is not n -small, but all of whose proper initial segments are n -small, if it exists and is unique.*

Definition 1.16. *Let $n \geq 1$, $x \in {}^\omega\omega$ and assume that $M_n^\#(x)$ exists. Then $M_n(x)$ is the unique x -premouse which is obtained from $M_n^\#(x)$ by iterating its top measure out of the universe.*

Remark. We denote $M_n^\#(0)$ and $M_n(0)$ by $M_n^\#$ and M_n for $n \geq 0$.

Remark. We say that $M_0^\#(x) = x^\#$ for all $x \in {}^\omega\omega$, where $x^\#$ denotes the least active ω_1 -iterable premouse if it exists. Moreover we say that $M_0(x) = L[x]$ is Gödel’s constructible universe above x .

The two correctness facts due to the third author about the premouse $M_n^\#(x)$ which are stated in the following lemma are going to be useful later, because

they help transferring projective statements from $M_n^\#(x)$ to V or the other way around.

Lemma 1.17. *Let $n \geq 0$ and assume that $M_n^\#(x)$ exists and is ω_1 -iterable for all $x \in {}^\omega\omega$. Let φ be a Σ_{n+2}^1 -formula.*

(1) *Assume n is even and let $x \in {}^\omega\omega$. Then we have for every real a in $M_n^\#(x)$,*

$$\varphi(a) \leftrightarrow M_n^\#(x) \models \varphi(a).$$

That means $M_n^\#(x)$ is Σ_{n+2}^1 -correct in V .

(2) *Assume n is odd, so in particular $n \geq 1$, and let $x \in {}^\omega\omega$. Then we have for every real a in $M_n^\#(x)$,*

$$\varphi(a) \leftrightarrow \Vdash_{\text{Col}(\omega, \delta_0)}^{M_n^\#(x)} \varphi(a),$$

where δ_0 denotes the least Woodin cardinal in $M_n^\#(x)$. Furthermore we have that $M_n^\#(x)$ is Σ_{n+1}^1 -correct in V .

For notational simplicity we sometimes just write a for the standard name \check{a} for a real a in $M_n^\#(x)$.

Proof of Lemma 1.17. For $n = 0$ this lemma holds by Shoenfield's Absoluteness Theorem (see for example Theorem 13.15 in [Ka08]) applied to the model we obtain by iterating the top measure of the active premouse $M_0^\#(x)$ and its images until we obtain a model of height $\geq \omega_1^V$, because this model has the same reals as $M_0^\#(x)$.

We simultaneously prove that (1) and (2) hold for all $n \geq 1$ inductively. In fact we are proving a more general statement: We will show inductively that both (1) and (2) hold for all n -iterable x -premice for all reals x in the sense of Definition 1.1 in [Ne95] which have n Woodin cardinals which are countable in V instead of the concrete x -premouse $M_n^\#(x)$ as in the statement of Lemma 1.17. Therefore notice that we could replace the z -premouse $M_n^\#(z)$ in the following argument by any z -premouse N which is n -iterable and has n Woodin cardinals which are countable in V .

Proof of (2): We start with proving (2) in the statement of Lemma 1.17 for n , assuming inductively that (1) and (2) hold for all $m < n$. For the downward implication assume that n is odd and let φ be a Σ_{n+2}^1 -formula such that $\varphi(a)$ holds in V for a parameter $a \in M_n^\#(z) \cap {}^\omega\omega$ for a $z \in {}^\omega\omega$. That means

$$\varphi(a) \equiv \exists x \forall y \psi(x, y, a)$$

for a Σ_n^1 -formula $\psi(x, y, a)$. Now fix a real \bar{x} in V such that

$$V \models \forall y \psi(\bar{x}, y, a).$$

We aim to show that

$$\Vdash_{\text{Col}(\omega, \delta_0)}^{M_n^\#(z)} \varphi(a),$$

where δ_0 denotes the least Woodin cardinal in $M_n^\#(z)$.

We first use Corollary 1.8 in [Ne95] to make \bar{x} generic over an iterate M^* of $M_n^\#(z)$ for the collapse of the image of the bottom Woodin cardinal δ_0 in $M_n^\#(z)$. That means there is an iteration tree \mathcal{T} on the n -iterable z -premouse $M_n^\#(z)$ of limit length and a non-dropping branch b through \mathcal{T} such that if

$$i : M_n^\#(z) \rightarrow M^*$$

denotes the corresponding iteration embedding we have that M^* is $(n-1)$ -iterable and if g is $\text{Col}(\omega, i(\delta_0))$ -generic over M^* , then $\bar{x} \in M^*[g]$.

We have that $M^*[g]$ can be construed as an $(z \oplus \bar{x})$ -premouse satisfying the inductive hypothesis and if we construe $M^*[g]$ as an $(z \oplus \bar{x})$ -premouse we have that in fact $M^*[g] = M_{n-1}^\#(z \oplus \bar{x})$ (see for example [SchSt09] for the fine structural details). Therefore we have inductively that the premouse $M^*[g]$ is Σ_n^1 -correct in V (in fact it is even Σ_{n+1}^1 -correct in V , but this is not necessary here) and using downwards absoluteness for the Π_{n+1}^1 -formula “ $\forall y \psi(\bar{x}, y, a)$ ” it follows that

$$M^*[g] \models \forall y \psi(\bar{x}, y, a),$$

because $\bar{x}, a \in M^*[g]$. Since the forcing $\text{Col}(\omega, i(\delta_0))$ is homogeneous, we have that

$$\Vdash_{\text{Col}(\omega, i(\delta_0))}^{M^*} \exists x \forall y \psi(x, y, a),$$

and by elementarity of the iteration embedding $i : M_n^\#(z) \rightarrow M^*$ it follows that

$$\Vdash_{\text{Col}(\omega, \delta_0)}^{M_n^\#(z)} \exists x \forall y \psi(x, y, a),$$

as desired.

For the upward implication of (2) in the statement of Lemma 1.17 let n again be odd, let z be a real and assume that

$$\Vdash_{\text{Col}(\omega, \delta_0)}^{M_n^\#(z)} \varphi(a),$$

where as above $\varphi(a) \equiv \exists x \forall y \psi(x, y, a)$ is a Σ_{n+2}^1 -formula, $\psi(x, y, a)$ is a Σ_n^1 -formula and a is a real such that $a \in M_n^\#(z)$. Let g be $\text{Col}(\omega, \delta_0)$ -generic over $M_n^\#(z)$ and pick a real \bar{x} such that

$$M_n^\#(z)[g] \models \forall y \psi(\bar{x}, y, a).$$

Since $M_n^\#(z) | (\delta_0^+)^{M_n^\#(z)}$ is countable in V , we can in fact pick the generic g such that we have $g \in V$. Then we have that $\bar{x} \in V$. Similar as above $M_n^\#(z)[g]$ can be construed as a z^* -premouse for some real z^* which satisfies the inductive hypothesis for $n-1$, in fact we again have that $M_n^\#(z)[g] = M_{n-1}^\#(z^*)$ if $M_n^\#(z)[g]$ is construed as a z^* -premouse. Since $n-1$ is even and we inductively assume that (1) in the statement of Lemma 1.17 holds

for all $m < n$, it follows from (1) applied to the Π_{n+1}^1 -formula “ $\forall y \psi(\bar{x}, y, a)$ ” and the premouse $M_n^\#(z)[g]$ that

$$V \models \forall y \psi(\bar{x}, y, a),$$

and therefore

$$V \models \exists x \forall y \psi(x, y, a),$$

as desired.

The fact that in this situation $M_n^\#(z)$ is Σ_{n+1}^1 -correct in V also follows from the inductive hypothesis, because $n - 1$ is even and the inductive hypothesis for $n - 1$ can be applied to the premouse $M_n^\#(z)$.

Proof of (1): Now we turn to the proof of (1) in the statement of Lemma 1.17. Let n be even and assume inductively that (1) and (2) hold for all $m < n$. We again start with the proof of the downward implication, that means we want to prove that

$$M_n^\#(z) \models \varphi(a),$$

where as above φ is a Σ_{n+2}^1 -formula which holds in V for $a \in M_n^\#(z) \cap {}^\omega \omega$ and $z \in {}^\omega \omega$. That means we again have

$$\varphi(a) \equiv \exists x \forall y \psi(x, y, a)$$

for a Σ_n^1 -formula $\psi(x, y, a)$. Since n is even, it follows from Moschovakis' Second Periodicity Theorem that the pointclass $\Pi_{n+1}^1(a)$ has the uniformization property (see Theorem 6C.5 in [Mo09]), because Theorem 2.14 in [Ne02] (see also Corollary 2.2) yields that Π_n^1 determinacy holds from the hypothesis that $M_{n-1}^\#(z)$ exists for all reals z . Consider the $\Pi_{n+1}^1(a)$ -definable set

$$\{x \mid \forall y \psi(x, y, a)\}.$$

The uniformization property yields the existence of a real \bar{x} such that we have $\{\bar{x}\} \in \Pi_{n+1}^1(a)$ and

$$V \models \forall y \psi(\bar{x}, y, a).$$

So let ρ be a Π_{n+1}^1 -formula such that

$$x = \bar{x} \leftrightarrow \rho(x, a)$$

for all $x \in {}^\omega \omega$. That means we have

$$V \models \rho(\bar{x}, a) \wedge \forall y \psi(\bar{x}, y, a).$$

Now we use, as in the proof of (2) above, the n -iterability of $M_n^\#(z)$ and Corollary 1.8 in [Ne95] to make \bar{x} generic over an iterate M^* of $M_n^\#(z)$ for the collapse of the image of the bottom Woodin cardinal δ_0 in $M_n^\#(z)$. As in the proof of (2) this means that in fact there is an iteration embedding

$$i : M_n^\#(z) \rightarrow M^*$$

such that M^* is $(n-1)$ -iterable and if $g \in V$ is $\text{Col}(\omega, i(\delta_0))$ -generic over M^* then $\bar{x} \in M^*[g]$. Since a is a real in $M_n^\#(z)$ we have that

$$\Vdash_{\text{Col}(\omega, \bar{\delta})}^{M^*[g]} \rho(\bar{x}, a) \wedge \forall y \psi(\bar{x}, y, a),$$

where $\bar{\delta}$ denotes the least Woodin cardinal inside $M^*[g]$, by the inductive hypothesis applied to the premouse $M^*[g]$, construed as an $(z \oplus \bar{x})$ -premouse, and the Π_{n+1}^1 -formula “ $\rho(\bar{x}, a) \wedge \forall y \psi(\bar{x}, y, a)$ ”, because $n-1$ is odd. As above we have that $M^*[g]$, construed as a $(z \oplus \bar{x})$ -premouse, satisfies the inductive hypothesis. Moreover we have as above that the inductive hypothesis applied to the model $M^*[g]$ and the Π_{n+1}^1 -formula “ $\rho(x, a) \wedge \forall y \psi(x, y, a)$ ” yields that in fact for all $x^* \in M^*[g]$

$$\Vdash_{\text{Col}(\omega, \bar{\delta})}^{M^*[g]} \rho(x^*, a) \wedge \forall y \psi(x^*, y, a) \text{ iff } V \models \rho(x^*, a) \wedge \forall y \psi(x^*, y, a).$$

By the homogeneity of the forcing $\text{Col}(\omega, i(\delta_0))$ this implies that the witness \bar{x} for x with $\rho(x, a)$ already exists in the ground model M^* , since $a \in M^*$ and \bar{x} is still the unique witness to the fact that the statement “ $\Vdash_{\text{Col}(\omega, \bar{\delta})}^{M^*[g]} \rho(\bar{x}, a) \wedge \forall y \psi(\bar{x}, y, a)$ ” holds true. Therefore it follows by downward absoluteness that

$$M^* \models \rho(\bar{x}, a) \wedge \forall y \psi(\bar{x}, y, a).$$

This implies that in particular

$$M^* \models \exists x \forall y \psi(x, y, a).$$

Using the elementarity of the iteration embedding we finally get that

$$M_n^\#(z) \models \exists x \forall y \psi(x, y, a).$$

For the proof of the upward implication in (1) let n again be even and assume that $M_n^\#(z) \models \exists x \forall y \psi(x, y, a)$ for $z \in {}^\omega\omega$ and a fixed real $a \in M_n^\#(z)$. Furthermore fix a real $\bar{x} \in M_n^\#(z)$ such that

$$M_n^\#(z) \models \forall y \psi(\bar{x}, y, a).$$

Then we obviously have that $\bar{x} \in V$. We want to show that $V \models \forall y \psi(\bar{x}, y, a)$. Assume not. That means

$$V \models \exists y \neg \psi(\bar{x}, y, a),$$

where “ $\exists y \neg \psi(\bar{x}, y, a)$ ” is a Σ_{n+1}^1 -formula. Therefore the downward implication we already proved applied to the formula “ $\exists y \neg \psi(\bar{x}, y, a)$ ” and the parameters $\bar{x}, a \in M_n^\#(z)$ yields that

$$M_n^\#(z) \models \exists y \neg \psi(\bar{x}, y, a),$$

which is a contradiction. \square

The proof of Lemma 1.17 with Shoenfield absoluteness replaced by Σ_1^1 absoluteness immediately gives the following lemma.

Lemma 1.18. *Let $n \geq 0$ and let M be a countable x -premouse with n Woodin cardinals for some $x \in {}^\omega\omega$ such that $M \models \text{ZFC}^-$ and M is ω_1 -iterable. Let φ be a Σ_{n+1}^1 -formula.*

(1) *Assume n is even. Then we have for every real a in M ,*

$$\varphi(a) \leftrightarrow M \models \varphi(a).$$

That means M is Σ_{n+1}^1 -correct in V .

(2) *Assume n is odd, so in particular $n \geq 1$. Then we have for every real a in M ,*

$$\varphi(a) \leftrightarrow \Vdash_{\text{Col}(\omega, \delta_0)}^M \varphi(a),$$

where δ_0 denotes the least Woodin cardinal in M . Furthermore we have that M is Σ_n^1 -correct in V .

The following lemma shows that Lemma 1.17 (1) does not hold if n is odd. Therefore the periodicity in the statement of Lemma 1.17 is necessary indeed.

Lemma 1.19. *Let $n \geq 1$ be odd and assume that $M_n^\#(x)$ exists and is ω_1 -iterable for all $x \in {}^\omega\omega$. Then $M_n^\#(x)$ is not Σ_{n+2}^1 -correct in V .*

Proof sketch. Consider for example the following Σ_{n+2}^1 -formula φ , where Π_{n+1}^1 -iterability is defined in Definition 1.6 in [St95] (see also Section 2.2 in this paper for some results related to Π_{n+1}^1 -iterability).

$$\begin{aligned} \varphi(x) \equiv \exists N \text{ such that } N \text{ is a countable } x\text{-premouse} \\ \text{which is } \Pi_{n+1}^1\text{-iterable and not } n\text{-small.} \end{aligned}$$

The statement “ N is Π_{n+1}^1 -iterable” is Π_{n+1}^1 -definable uniformly in any code for N (see Lemma 1.7 in [St95]). Therefore φ is a Σ_{n+2}^1 -formula.

We have that $\varphi(x)$ holds in V for all reals x as witnessed by the x -premouse $M_n^\#(x)$, because ω_1 -iterability implies Π_{n+1}^1 -iterability, since we assumed that $M_n^\#(x)$ exists for all $x \in {}^\omega\omega$ (see Lemma 2.12 (2) which uses Lemma 2.2 in [St95]).

Assume toward a contradiction that $\varphi(x)$ holds in $M_n^\#(x)$ as witnessed by some x -premouse N in $M_n^\#(x)$ which is Π_{n+1}^1 -iterable and not n -small. Since n is odd, Lemma 3.1 in [St95] implies that $\mathbb{R} \cap M_n(x) \subseteq \mathbb{R} \cap N$, which is a contradiction as $N \in M_n^\#(x)$.

Therefore $\varphi(x)$ cannot hold in $M_n^\#(x)$ and thus $M_n^\#(x)$ is not Σ_{n+2}^1 -correct in V . \square

See for example [St95] for further information on the premouse $M_n^\#$.

1.4. Mice and Determinacy. Some of the results mentioned in Section 1.1 can be improved using the existence of certain mice instead of large cardinals in V as an hypothesis. In this section we will list results of that form and mention some things known about the converse direction.

In the context of analytic determinacy Harrington was able to prove the converse of Martin's result from [Ma70] and therefore obtained the following theorem.

Theorem 1.20 (Harrington in [Ha78], Martin in [Ma70]). *The following are equivalent over ZFC.*

- (i) *The mouse $0^\#$ exists, and*
- (ii) *every Π_1^1 -definable set of reals is determined.*

In the projective hierarchy Neeman improved the result of [MaSt89] as follows. Here “ \mathcal{D} ” denotes the game quantifier as also used in Section 3.5 later (see Section 6D in [Mo09] for a definition and some basic facts about the game quantifier “ \mathcal{D} ”).

Theorem 1.21 (Neeman in [Ne02]). *Let $n \geq 1$ and assume that $M_n^\#$ exists and is ω_1 -iterable. Then every $\mathcal{D}^{(n)}(< \omega^2 - \Pi_1^1)$ -definable set of reals is determined and thus in particular every Π_{n+1}^1 -definable set of reals is determined.*

In this paper we will present a proof of the boldface version of a converse direction of this theorem due to the third author which is only assuming that every Π_{n+1}^1 -definable set of reals is determined, see Corollary 2.2. The lightface version of the analogous converse direction of Theorem 1.21 is still open for $n > 1$ (see also Section 4.2). For $n = 1$ it is due to the third author, who proved the following theorem (see Corollary 4.17 in [StW16]).

Theorem 1.22. *The following are equivalent over ZFC.*

- (i) *$M_1^\#$ exists and is ω_1 -iterable, and*
- (ii) *every Π_1^1 -definable and every Δ_2^1 -definable set of reals is determined.*

Here the implication (i) \Rightarrow (ii), which also follows from Theorem 2.14 in [Ne02] (see Theorem 1.21 above), was first shown by the third author in unpublished work.

At the level of infinitely many Woodin cardinals we have that the following equivalence, which is also due to the third author, holds true (see Theorem 8.4 in [KW10]).

Theorem 1.23 ([KW10]). *The following are equivalent over ZFC.*

- (i) *$M_\omega^\#$ exists and is countably iterable, and*
- (ii) *$\text{AD}^{L(\mathbb{R})}$ holds and $\mathbb{R}^\#$ exists.*

Here we mean by “ $\text{AD}^{L(\mathbb{R})}$ holds” that every set of reals in $L(\mathbb{R})$ is determined.

2. A MODEL WITH WOODIN CARDINALS FROM DETERMINACY HYPOTHESES

In this section we are going to construct a proper class model with n Woodin cardinals from $\mathbf{\Pi}_n^1$ determinacy together with $\mathbf{\Pi}_{n+1}^1$ determinacy, but the model constructed in this section need not be iterable. We will treat iterability issues for models like the one constructed in this section later in Section 3.

2.1. Introduction. The main goal of Sections 2 and 3 is to give a proof of the following theorem due to the third author.

Theorem 2.1. *Let $n \geq 1$ and assume $\mathbf{\Pi}_{n+1}^1$ determinacy holds. Then $M_n^\#(x)$ exists and is ω_1 -iterable for all $x \in {}^\omega\omega$.*

The converse of Theorem 2.1 also holds true. For odd n it is due to the third author in never-published work and for even n it is due to Itay Neeman in [Ne95]. This yields the following corollary, where the case $n = 0$ is due to D. A. Martin (see [Ma70]) and L. Harrington (see [Ha78]).

Corollary 2.2. *Let $n \geq 0$. Then the following are equivalent.*

- (1) $\mathbf{\Pi}_{n+1}^1$ determinacy, and
- (2) for all $x \in {}^\omega\omega$, $M_n^\#(x)$ exists and is ω_1 -iterable.

The proof of Theorem 2.1 is organized inductively. Thereby Harrington's result that analytic determinacy yields the existence of $0^\#$ (see [Ha78]) is the base step of our induction. So we will assume throughout the proof of Theorem 2.1 at the n 'th level, that Theorem 2.1 holds true at the level $n - 1$. In fact by Theorem 2.14 in [Ne02] we can assume during the proof at the level n that the existence and ω_1 -iterability of $M_{n-1}^\#(x)$ for all $x \in {}^\omega\omega$ is equivalent to $\mathbf{\Pi}_n^1$ determinacy (see Corollary 2.2, for odd n this result is due to the third author). We will use this in what follows without further notice.

We first fix some notation we are going to use for the rest of this paper.

If M is a premouse let δ_M denote the least Woodin cardinal in M , if it exists. For $n \geq 2$ and $x \in {}^\omega\omega$ let $\delta_x = \delta_{M_{n-1}(x)}$ denote the least Woodin cardinal in $M_{n-1}(x)$, whenever this does not lead to confusion. Moreover in case we are considering $L[x] = M_0(x)$ and a confusion is not possible let δ_x denote the least x -indiscernible in $L[x]$.

Remark. Recall that a real $x \in {}^\omega\omega$ is *Turing reducible* to a real $y \in {}^\omega\omega$ (write " $x \leq_T y$ ") iff x is recursive in y or equivalently iff there exists an oracle Turing machine that computes x using y as an oracle. Moreover we write $x \equiv_T y$ iff $x \leq_T y$ and $y \leq_T x$ and say in this case that x and y are *Turing equivalent*.

The following lemma generalizes a theorem of Kechris and Solovay (see Theorem 3.1 in [KS85]) to the context of mice with finitely many Woodin cardinals. It is one key ingredient for building inner models with finitely many Woodin cardinals from determinacy hypotheses and therefore in particular for proving Theorem 2.1. The following two sections will be devoted to the proof of this lemma.

Lemma 2.3. *Let $n \geq 1$. Assume that $M_{n-1}^\#(x)$ exists and is ω_1 -iterable for all $x \in {}^\omega\omega$ and that all Σ_{n+1}^1 -definable sets of reals are determined. Then there exists a real y_0 such that for all reals $x \geq_T y_0$,*

$$M_{n-1}(x)|\delta_x \models \text{OD-determinacy}.$$

The main difficulty in proving Lemma 2.3 in our context is the fact that for $n > 1$ the premouse $M_{n-1}(x)|\delta_x$ has lots of total extenders on its sequence. This is the main reason why we cannot generalize the proof of Theorem 3.1 in [KS85] straightforwardly. Therefore we need to prove some preliminary lemmas concerning comparisons and $L[E]$ -constructions in our context in the following section. Some of this could have been avoided, if we would only want to prove Lemma 2.3 for models like for example lower part models (see Definition 3.21), which do not contain total extenders on their sequence.

2.2. Preliminaries. In this section we prove a few general lemmas about $(n-1)$ -small premice which we are going to need for the proof of Lemma 2.3 and which are also going to be helpful later on.

The following models, called \mathcal{Q} -structures, can serve as witnesses for iterability by guiding an iteration strategy as in Definition 2.5. See also for example the proof of Lemma 2.11 for an application of this iteration strategy.

Informally a \mathcal{Q} -structure for a cofinal well-founded branch b through \mathcal{T} is the longest initial segment of $\mathcal{M}_b^\mathcal{T}$ at which $\delta(\mathcal{T})$ is still seen to be a Woodin cardinal. Such \mathcal{Q} -structures are also introduced for example in Definition 6.11 in [St10].

Definition 2.4. *Let N be an arbitrary premouse and let \mathcal{T} be an iteration tree on N of limit length.*

- (1) *We say $\mathcal{Q} = \mathcal{Q}(\mathcal{T})$ is a \mathcal{Q} -structure for \mathcal{T} iff $\mathcal{M}(\mathcal{T}) \trianglelefteq \mathcal{Q}$, $\delta(\mathcal{T})$ is a cutpoint of \mathcal{Q} , \mathcal{Q} is ω_1 -iterable above $\delta(\mathcal{T})$,*

$$\mathcal{Q} \models \text{“}\delta(\mathcal{T}) \text{ is a Woodin cardinal”},$$

if $\mathcal{Q} \neq \mathcal{M}(\mathcal{T})$ and either

- (i) *over \mathcal{Q} there exists an $r\Sigma_n$ -definable set $A \subset \delta(\mathcal{T})$ such that there is no $\kappa < \delta(\mathcal{T})$ such that κ is strong up to $\delta(\mathcal{T})$ with respect to A as being witnessed by extenders on the sequence of \mathcal{Q} for some $n < \omega$, or*
- (ii) *$\rho_n(\mathcal{Q}) < \delta(\mathcal{T})$ for some $n < \omega$.*

(2) Let b be a cofinal well-founded branch through \mathcal{T} . Then we say $\mathcal{Q} = \mathcal{Q}(b, \mathcal{T})$ is a \mathcal{Q} -structure for b in \mathcal{T} iff $\mathcal{Q} = \mathcal{M}_b^{\mathcal{T}}|_{\gamma}$, where $\gamma \leq \mathcal{M}_b^{\mathcal{T}} \cap \text{Ord}$ is the least ordinal such that either

$$\gamma < \mathcal{M}_b^{\mathcal{T}} \cap \text{Ord} \text{ and } \mathcal{M}_b^{\mathcal{T}}|_{(\gamma+1)} \models \text{“}\delta(\mathcal{T}) \text{ is not Woodin”},$$

or

$$\gamma = \mathcal{M}_b^{\mathcal{T}} \cap \text{Ord} \text{ and } \rho_n(\mathcal{M}_b^{\mathcal{T}}) < \delta(\mathcal{T})$$

for some $n < \omega$ or over $\mathcal{M}_b^{\mathcal{T}}$ there exists an $r\Sigma_n$ -definable set $A \subset \delta(\mathcal{T})$ such that there is no $\kappa < \delta(\mathcal{T})$ such that κ is strong up to $\delta(\mathcal{T})$ with respect to A as being witnessed by extenders on the sequence of $\mathcal{M}_b^{\mathcal{T}}$ for some $n < \omega$.

If no such ordinal $\gamma \leq \mathcal{M}_b^{\mathcal{T}} \cap \text{Ord}$ exists, we let $\mathcal{Q}(b, \mathcal{T})$ be undefined.

For the notion of an $r\Sigma_n$ -definable set see for example §2 in [MS94].

Remark. We are also going to use the notion of a Π_n^1 -iterable \mathcal{Q} -structure $\mathcal{Q}(\mathcal{T})$, meaning that $\mathcal{Q}(\mathcal{T})$ is Π_n^1 -iterable above $\delta(\mathcal{T})$, $\delta(\mathcal{T})$ -solid and that $\mathcal{Q}(\mathcal{T})$ satisfies all properties in (1) except for the ω_1 -iterability above $\delta(\mathcal{T})$. It will be clear from the context if we include ω_1 -iterability in the definition of \mathcal{Q} -structure or not. Here Π_n^1 -iterability is defined as in Definitions 1.4 and 1.6 in [St95] (see also the explanations before Lemma 2.12).

Remark. In Case (1)(i) in Definition 2.4 we have that in particular $\rho_n(\mathcal{Q}) \leq \delta$ and if $\mathcal{Q} \triangleleft M$ for a premouse M and if we let $\gamma = \mathcal{Q} \cap \text{Ord}$, then we have that $\delta(\mathcal{T})$ is not Woodin in $J_{\gamma+1}^M$. The same thing for M as above holds true in Case (1)(ii) in Definition 2.4, because in this case $\delta(\mathcal{T})$ is not even a cardinal in $J_{\gamma+1}^M$.

Remark. Let $n \geq 1$ and assume that $M_n^{\#}(x)$ exists and is ω_1 -iterable for all reals x . Then any \mathcal{Q} -structure \mathcal{Q} for an iteration tree \mathcal{T} on an n -small premouse is unique by a comparison argument as the one we will see in the proof of Lemma 2.11.

Definition 2.5. Let N be a premouse. Then a possibly partial iteration strategy Σ for N is the \mathcal{Q} -structure iteration strategy for N or we say that Σ is guided by \mathcal{Q} -structures, iff Σ is defined as follows. For a tree \mathcal{U} on N of limit length and a branch b through \mathcal{U} we let

$$\Sigma(\mathcal{U}) = b \text{ iff } \mathcal{Q}(\mathcal{U}) \text{ exists and } \mathcal{Q}(b, \mathcal{U}) = \mathcal{Q}(\mathcal{U}),$$

if such a branch b exists and is unique. If no such unique branch b through \mathcal{U} exists, we let $\Sigma(\mathcal{U})$ be undefined.

Lemma 2.6. Let $n \geq 1$ and assume that $M_n^{\#}(x)$ exists and is ω_1 -iterable for all reals x . Let N be an n -small premouse and let \mathcal{T} be an iteration tree on N . Then the branch b through \mathcal{T} which satisfies $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$ as in the definition of the \mathcal{Q} -structure iteration strategy Σ is in fact unique.

This lemma holds true because for two branches b and c through an iteration tree \mathcal{T} as above, we have that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(c, \mathcal{T})$ implies that $b = c$ by the Branch Uniqueness Theorem (see Theorem 6.10 in [St10]). This is proven in Corollary 6 in §6 of [Je97] (written by Martin Zeman).

Remark. Let $n \geq 1$ and assume that $M_n^\#(x)$ exists and is ω_1 -iterable for all reals x . Let N be an n -small premouse and let \mathcal{T} be an iteration tree on N . Then the branch $b = \Sigma(\mathcal{T})$ given by the \mathcal{Q} -structure iteration strategy is in fact cofinal.

The following notion will be important in what follows to ensure that \mathcal{Q} -structures exist.

Definition 2.7. *Let M be a premouse and let δ be a cardinal in M or $\delta = M \cap \text{Ord}$. We say that δ is not definably Woodin over M iff there exists an ordinal $\gamma \leq M \cap \text{Ord}$ such that $\gamma \geq \delta$ and either*

- (i) *over J_γ^M there exists an $r\Sigma_n$ -definable set $A \subset \delta$ such that there is no $\kappa < \delta$ such that κ is strong up to δ with respect to A as witnessed by extenders on the sequence of M for some $n < \omega$, or*
- (ii) *$\rho_n(J_\gamma^M) < \delta$ for some $n < \omega$.*

For several iterability arguments to follow we need our premice to satisfy the following property. By a fine structural argument this property is preserved during an iteration and will therefore ensure that \mathcal{Q} -structures exist in an iteration of a premouse M satisfying this property.

Definition 2.8. *Let M be a premouse. We say M has no definable Woodin cardinals iff for all $\delta \leq M \cap \text{Ord}$ we have that δ is not definably Woodin over M .*

Remark. Let M be a premouse which has no definable Woodin cardinals. Note that M might still have Woodin cardinals. Consider for example the premouse $M_1^\#$, which by definition has a Woodin cardinal, but no definable Woodin cardinals since $\rho_\omega(M_1^\#) = \omega$.

In what follows we sometimes want to consider premice which are obtained from $M_n^\#$ “constructed on top” of a premouse N . The following definition makes precise what we mean by that.

Definition 2.9. *Let $n \geq 1$ and assume that $M_n^\#(x)$ exists for all reals x . Let N be a countable x -premouse for some $x \in {}^\omega\omega$. Then we say $M_n^\#(N)$ is the smallest x -premouse $M \supseteq N$ with*

$$\rho_\omega(M) \leq N \cap \text{Ord}$$

which is ω_1 -iterable above $N \cap \text{Ord}$, sound above $N \cap \text{Ord}$ and such that either M is not fully sound, or M is not n -small above $N \cap \text{Ord}$.

In the first case, i.e. if M is not fully sound, we sometimes say that the construction of $M_n^\#(N)$ breaks down.

Remark. We can define a premouse $M_n(N)$ in a similar fashion, by iterating the top extender of $M = M_n^\#(N)$ out of the universe in the case that M is not n -small above $N \cap \text{Ord}$. In the case that M is not fully sound, we just let $M_n(N) = M$. So in particular $M_n(N)$ is a proper class model in the first case and a set in the latter case.

In what follows we will point out if $M_n^\#(N)$ denotes the premouse constructed in the sense of Definition 2.9 or if it denotes the premouse $M_n^\#(x)$ in the usual sense, where x is for example a real coding the countable premouse N . Note that these two notions are different since extenders on the N -sequence are included in the $M_n^\#(N)$ -sequence if $M_n^\#(N)$ is constructed in the sense of Definition 2.9.

We will need the following notation.

- Definition 2.10.** (1) Let $x, y \in {}^\omega\omega$ be such that $x = (x_n \mid n < \omega)$ for $x_n \in \omega$ and $y = (y_n \mid n < \omega)$ for $y_n \in \omega$. Then we let $x \oplus y = (x_0, y_0, x_1, y_1, \dots) \in {}^\omega\omega$.
- (2) Let M and N be countable premice. We say a real x codes $M \oplus N$ iff $x \geq_T x_M \oplus x_N$ for a real x_M coding M and a real x_N coding N .

The following lemma proves that under the right hypothesis comparison works for certain ω_1 -iterable premice instead of $(\omega_1 + 1)$ -iterable premice as in the usual Comparison Lemma (see Theorem 3.11 in [St10]). Moreover the proof of this lemma will use arguments that are explained here in full detail and will show up in several different proofs throughout this paper again with possibly less details given.

Recall that δ_x denotes the least Woodin cardinal in $M_{n-1}^\#(x)$ for $n \geq 2$.

Lemma 2.11. Let $n \geq 1$ and assume that $M_{n-1}^\#(x)$ exists and is ω_1 -iterable for all reals x . Let M and N be countable premice, such that M and N have a common cutpoint δ . Assume that M and N both do not have definable Woodin cardinals above δ and that every proper initial segment of M and N is $(n-1)$ -small above δ .

- (1) Let x be an arbitrary real and let $n \geq 2$. Then $H_{\delta_x}^{M_{n-1}^\#(x)}$ is closed under the operation

$$a \mapsto M_{n-2}^\#(a)$$

and moreover this operation $a \mapsto M_{n-2}^\#(a)$ for $a \in H_{\delta_x}^{M_{n-1}^\#(x)}$ is contained in $M_{n-1}^\#(x)|\delta_x$.

- (2) Let x be a real coding M and assume that the premouse M is ω_1 -iterable above δ . If Σ denotes the ω_1 -iteration strategy for M above δ , then

$$\Sigma \upharpoonright H_{\delta_x}^{M_{n-1}^\#(x)} \in M_{n-1}^\#(x)|\delta_x.$$

- (3) Let x be a real coding $M \oplus N$ and assume that the premice M and N are both ω_1 -iterable above δ . Moreover assume that

$$M|\delta = N|\delta.$$

Then we can successfully coiterate M and N above δ inside the model $M_{n-1}^\#(x)$. That means there are iterates M^* of M and N^* of N above δ such that we have

- (a) $M^* \leq N^*$ and the iteration from M to M^* does not drop, or
(b) $N^* \leq M^*$ and the iteration from N to N^* does not drop.

In particular the coiteration is successful in V in the same sense.

- (4) Let x be a real coding $M \oplus N$ and assume that the premice M and N are both ω_1 -iterable above δ . Moreover assume that

$$M|\delta = N|\delta,$$

M and N are δ -sound, $\rho_\omega(M) \leq \delta$ and $\rho_\omega(N) \leq \delta$. Then we have

$$M \leq N \text{ or } N \leq M.$$

Remark. This lemma holds in particular for $\delta = \omega$. That means if we assume that $M_{n-1}^\#(x)$ exists and is ω_1 -iterable for all reals x and M and N are ω_1 -iterable countable premice such that both do not have definable Woodin cardinals and such that every proper initial segment of M and N is $(n-1)$ -small, then we can successfully compare M and N as in Lemma 2.11 (3).

Proof of Lemma 2.11. Proof of (1): Let $a \in H_{\delta_x}^{M_{n-1}^\#(x)}$ be arbitrary and perform a fully backgrounded extender construction $L[E](a)^{M_{n-1}^\#(x)|\kappa}$ in the sense of [MS94] (with the smallness hypothesis weakened to allow ω -small premice in the construction) above a inside the model $M_{n-1}^\#(x)|\kappa$, where κ denotes the critical point of the top measure of the active premouse $M_{n-1}^\#(x)$. Then we have that the premouse $L[E](a)^{M_{n-1}^\#(x)|\kappa}$ has $n-1$ Woodin cardinals by a generalization of Theorem 11.3 in [MS94]. So in particular it follows that $L[E](a)^{M_{n-1}^\#(x)|\kappa}$ is not $(n-2)$ -small. Moreover the premouse $L[E](a)^{M_{n-1}^\#(x)|\kappa}$ inherits the iterability from $M_{n-1}^\#(x)$ and therefore we have that

$$M_{n-2}^\#(a) \triangleleft L[E](a)^{M_{n-1}^\#(x)|\kappa}.$$

In fact the operation $a \mapsto M_{n-2}^\#(a)$ for $a \in H_{\delta_x}^{M_{n-1}^\#(x)}$ is contained in $M_{n-1}^\#(x)|\delta_x$, because $M_{n-2}^\#(a)$ can be obtained from an $L[E]$ -construction above a .

Proof of (2) + (3) + (4): We prove (2),(3) and (4) simultaneously by an inductive argument. For $n=1$ there is nothing to show, because we defined M to be 0-small iff M is an initial segment of Gödel's constructible universe L . That means if M and N are such that every proper initial segment

of M or N is 0-small above some common cutpoint δ as in the statement of Lemma 2.11, then every iteration tree on M or N above δ is linear and there is nothing to show for (2). Moreover we easily get that one is an initial segment of the other since every proper initial segment of M or N is above δ as an initial segment of L . Therefore (3) and (4) hold.

So let $n \geq 2$ and assume that (2),(3) and (4) hold for $n-2$. We first want to show that (2) holds for $n-1$. Let us assume for notational simplicity that $\delta = \omega$ and let M be an ω_1 -iterable premouse, such that every proper initial segment of M is $(n-1)$ -small and such that M has no definable Woodin cardinals. Let x be a real coding the premouse M and assume that $M_{n-1}^\#(x)$ exists and is ω_1 -iterable. Further let Σ be the ω_1 -iteration strategy for M and let \mathcal{T} be an iteration tree on M in V of length $\lambda + 1$ for some limit ordinal $\lambda < \omega_1^V$ such that \mathcal{T} is according to Σ and

$$\mathcal{T} \upharpoonright \lambda \in H_{\delta_x}^{M_{n-1}^\#(x)}.$$

By assumption the premouse M is ω_1 -iterable in V and has no definable Woodin cardinals. Therefore Σ is the \mathcal{Q} -structure iteration strategy (see Definition 2.5) and there exists a \mathcal{Q} -structure $\mathcal{Q} \trianglelefteq \mathcal{M}_\lambda^\mathcal{T}$ for $\mathcal{T} \upharpoonright \lambda$ which is ω_1 -iterable in V . We first want to show that such an ω_1 -iterable \mathcal{Q} -structure already exists in the model $M_{n-1}^\#(x)|\delta_x$.

First consider the case

$$\mathcal{Q} = \mathcal{M}(\mathcal{T} \upharpoonright \lambda),$$

where the latter denotes the common part model of $\mathcal{T} \upharpoonright \lambda$. In this case \mathcal{Q} is also a \mathcal{Q} -structure for $\mathcal{T} \upharpoonright \lambda$ inside the model $M_{n-1}^\#(x)|\delta_x$ for trivial reasons, because the condition that \mathcal{Q} needs to be ω_1 -iterable above $\delta(\mathcal{T} \upharpoonright \lambda) = \mathcal{M}(\mathcal{T} \upharpoonright \lambda) \cap \text{Ord}$ is empty here and therefore \mathcal{Q} can be isolated as the \mathcal{Q} -structure for $\mathcal{T} \upharpoonright \lambda$ in $M_{n-1}^\#(x)|\delta_x$.

So we can assume now that

$$\mathcal{M}(\mathcal{T} \upharpoonright \lambda) \triangleleft \mathcal{Q}.$$

That means $\delta(\mathcal{T} \upharpoonright \lambda) \in \mathcal{Q}$ and therefore \mathcal{Q} is by definition the longest initial segment of $\mathcal{M}_\lambda^\mathcal{T}$ in V such that

$$\mathcal{Q} \models \text{“}\delta(\mathcal{T} \upharpoonright \lambda) \text{ is Woodin”}.$$

Every proper initial segment of $\mathcal{M}_\lambda^\mathcal{T}$ is $(n-1)$ -small since the same holds for M . Thus every proper initial segment of \mathcal{Q} is $(n-1)$ -small. Together with the fact that

$$\mathcal{Q} \models \text{“}\delta(\mathcal{T} \upharpoonright \lambda) \text{ is Woodin”},$$

this implies that every proper initial segment of \mathcal{Q} in fact has to be $(n-2)$ -small above $\delta(\mathcal{T} \upharpoonright \lambda)$.

Now consider the premouse $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ in the sense of Definition 2.9, which exists inside $M_{n-1}^\#(x)|\delta_x$ because of (1), since we have that $\mathcal{T} \upharpoonright \lambda \in$

$H_{\delta_x}^{M_{n-1}^\#(x)}$. Note that all proper initial segments of $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ are $(n-2)$ -small above

$$\delta(\mathcal{T} \upharpoonright \lambda) = \mathcal{M}(\mathcal{T} \upharpoonright \lambda) \cap \text{Ord},$$

regardless of whether the case that $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ is not fully sound or the case that $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ is not $(n-2)$ -small above $\delta(\mathcal{T} \upharpoonright \lambda)$ in the definition of $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ (see Definition 2.9) holds. Moreover we have by definition that

$$\mathcal{Q} \upharpoonright \delta(\mathcal{T} \upharpoonright \lambda) = \mathcal{M}(\mathcal{T} \upharpoonright \lambda) = M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda)) \upharpoonright \delta(\mathcal{T} \upharpoonright \lambda).$$

Thus a coiteration of \mathcal{Q} and $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ would take place above $\delta(\mathcal{T} \upharpoonright \lambda)$. Moreover $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ and \mathcal{Q} are both $\delta(\mathcal{T} \upharpoonright \lambda)$ -sound and project to $\delta(\mathcal{T} \upharpoonright \lambda)$, so in particular they both do not have definable Woodin cardinals above $\delta(\mathcal{T} \upharpoonright \lambda)$. This implies by the inductive hypothesis (4) that the comparison of \mathcal{Q} and $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ is successful in V , because all proper initial segments of both \mathcal{Q} and $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ are $(n-2)$ -small above $\delta(\mathcal{T} \upharpoonright \lambda)$. Moreover both sides do not move in the comparison as in (4) and therefore we can distinguish two cases as follows.

Case 1. Assume that

$$M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda)) \triangleleft \mathcal{Q}.$$

Then we have that

$$M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda)) \models \text{“}\delta(\mathcal{T} \upharpoonright \lambda) \text{ is Woodin”},$$

because we have by definition of the \mathcal{Q} -structure \mathcal{Q} (see Definition 2.4) that

$$\mathcal{Q} \models \text{“}\delta(\mathcal{T} \upharpoonright \lambda) \text{ is Woodin”}.$$

Case 1.1. Assume that the premouse $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ is not $(n-2)$ -small above $\mathcal{M}(\mathcal{T} \upharpoonright \lambda) \cap \text{Ord}$.

That means we were able to construct the full premouse $M_{n-2}^\#$ on top of $\mathcal{M}(\mathcal{T} \upharpoonright \lambda)$ as in the sense of Definition 2.9. In this case $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ is not $(n-1)$ -small, because $\delta(\mathcal{T} \upharpoonright \lambda)$ is a Woodin cardinal in $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$. But we argued earlier that every proper initial segment of \mathcal{Q} is $(n-1)$ -small, which contradicts $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda)) \triangleleft \mathcal{Q}$.

Case 1.2. Assume now that we are in the other case of Definition 2.9. That means we have that $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ is not fully sound.

So $\rho_m(M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))) < \delta(\mathcal{T} \upharpoonright \lambda)$ for some $m < \omega$. This implies that $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ is already a \mathcal{Q} -structure for $\mathcal{T} \upharpoonright \lambda$ inside the model

$M_{n-1}^\#(x)|\delta_x$, because $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ exists inside $M_{n-1}^\#(x)|\delta_x$ by part (1) of this lemma as argued earlier and is by definition ω_1 -iterable above $\delta(\mathcal{T} \upharpoonright \lambda)$, which finishes the argument for this case.

Case 2. Assume that

$$\mathcal{Q} \trianglelefteq M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda)).$$

This implies that \mathcal{Q} is in $M_{n-1}^\#(x)|\delta_x$ and furthermore is ω_1 -iterable above $\delta(\mathcal{T} \upharpoonright \lambda)$ in $M_{n-1}^\#(x)|\delta_x$ since the same holds for $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ by part (1) of this lemma.

So we showed that in both cases there exists an ω_1 -iterable \mathcal{Q} -structure \mathcal{Q} for $\mathcal{T} \upharpoonright \lambda$ in $M_{n-1}^\#(x)|\delta_x$. We now aim to show that the cofinal well-founded branch through \mathcal{T} in V which is given by the \mathcal{Q} -structure iteration strategy Σ , that means the branch b for which we have

$$\mathcal{Q}(b, \mathcal{T} \upharpoonright \lambda) = \mathcal{Q},$$

is also contained in $M_{n-1}^\#(x)|\delta_x$.

Consider the statement

$$\begin{aligned} \phi(\mathcal{T} \upharpoonright \lambda, \mathcal{Q}) \equiv & \text{“there is a cofinal branch } b \text{ through } \mathcal{T} \upharpoonright \lambda \text{ such that} \\ & \text{there is a } \mathcal{Q}^* \trianglelefteq \mathcal{M}_b^{\mathcal{T}} \text{ with } \mathcal{Q}^* \cong \mathcal{Q}\text{”}. \end{aligned}$$

This statement is Σ_1^1 -definable uniformly in codes for $\mathcal{T} \upharpoonright \lambda$ and \mathcal{Q} and obviously true in V as witnessed by the branch b given by the ω_1 -iterability of the premouse M . Since the iteration tree $\mathcal{T} \upharpoonright \lambda \in H_{\delta_x}^{M_{n-1}^\#(x)}$ need not be countable in the model $M_{n-1}^\#(x)|\delta_x$, we consider the model $(M_{n-1}^\#(x)|\delta_x)^{\text{Col}(\omega, \gamma)}$ instead, where $\gamma < \delta_x$ is an ordinal such that

$$\mathcal{T} \upharpoonright \lambda, \mathcal{Q} \in (M_{n-1}^\#(x)|\delta_x)^{\text{Col}(\omega, \gamma)}$$

are countable inside the model $(M_{n-1}^\#(x)|\delta_x)^{\text{Col}(\omega, \gamma)}$, where with the model $(M_{n-1}^\#(x)|\delta_x)^{\text{Col}(\omega, \gamma)}$ we denote an arbitrary $\text{Col}(\omega, \gamma)$ -generic extension of the model $M_{n-1}^\#(x)|\delta_x$.

Then it follows by Σ_1^1 -absoluteness that the statement $\phi(\mathcal{T} \upharpoonright \lambda, \mathcal{Q})$ holds in $(M_{n-1}^\#(x)|\delta_x)^{\text{Col}(\omega, \gamma)}$ as witnessed by some branch \bar{b} and some model $\bar{\mathcal{Q}}$.

Since by the argument above \mathcal{Q} is a \mathcal{Q} -structure for \mathcal{T} in $M_{n-1}^\#(x)|\delta_x$, we have that $\mathcal{Q} = \bar{\mathcal{Q}}$ and it follows that \bar{b} is the unique cofinal branch through \mathcal{T} with $\mathcal{Q}(\bar{b}, \mathcal{T} \upharpoonright \lambda) = \mathcal{Q}$ by Lemma 2.6.

Since the branch \bar{b} is uniquely definable from $\mathcal{T} \upharpoonright \lambda$ and \mathcal{Q} , and we have that $\mathcal{T} \upharpoonright \lambda, \mathcal{Q} \in M_{n-1}^\#(x)|\delta_x$, it follows by homogeneity of the forcing $\text{Col}(\omega, \gamma)$ that actually $\bar{b} \in M_{n-1}^\#(x)|\delta_x$.

Thus we have that $\Sigma(\mathcal{T}) = \bar{b} \in M_{n-1}^\#(x)|\delta_x$ and our argument shows that in fact the operation $\mathcal{T} \mapsto \Sigma(\mathcal{T})$ for iteration trees $\mathcal{T} \in H_{\delta_x}^{M_{n-1}^\#(x)}$ on M of limit length is in the model $M_{n-1}^\#(x)|\delta_x$ for the following reason. Let \mathcal{T} be an iteration tree on M of limit length such that $\mathcal{T} \in H_{\delta_x}^{M_{n-1}^\#(x)}$. Then we showed in the first part of this proof that $M_{n-1}^\#(x)|\delta_x$ can find a \mathcal{Q} -structure \mathcal{Q} for \mathcal{T} . Now $M_{n-1}^\#(x)|\delta_x$ can compute $\Sigma(\mathcal{T})$ from \mathcal{T} , because $\Sigma(\mathcal{T})$ is the unique cofinal branch b through \mathcal{T} such that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}$ and we showed that this branch b exists inside $M_{n-1}^\#(x)|\delta_x$. Therefore we proved that (2) holds.

To show (3) assume now in addition that N is an ω_1 -iterable premouse such that every proper initial segment of N is $(n-1)$ -small and N has no definable Woodin cardinals. Moreover let x be a real coding $M \oplus N$. We again assume that $\delta = \omega$ for notational simplicity.

By our hypothesis we have that $M_{n-1}^\#(x)$ exists, so we work inside the model $M_{n-1}^\#(x)$. Moreover it follows from (2) that M and N are iterable inside $M_{n-1}^\#(x)$, particularly with respect to trees in $H_{\omega_2}^{M_{n-1}^\#(x)}$, since $M_{n-1}^\#(x)$ is countable in V and thus

$$\omega_2^{M_{n-1}^\#(x)} < \omega_1^V.$$

In particular M and N are $(\omega_1 + 1)$ -iterable in $M_{n-1}^\#(x)$ and the coiteration of M and N terminates successfully inside the model $M_{n-1}^\#(x)$ by the usual Comparison Lemma (see Theorem 3.11 in [St10]) applied inside $M_{n-1}^\#(x)$. This shows that (3) holds.

To prove (4) we assume that we moreover have that M and N are ω -sound and that $\rho_\omega(M) = \omega$ and $\rho_\omega(N) = \omega$. Then it follows as in Corollary 3.12 in [St10] that we have $M \trianglelefteq N$ or $N \trianglelefteq M$. \square

Remark. As in the previous lemma let $n \geq 1$ and assume that $M_{n-1}^\#(x)$ exists and is ω_1 -iterable for all reals x . Then we can also successfully coiterate countable ω_1 -iterable premice M and N which agree up to a common cutpoint δ and are $(n-1)$ -small above δ in the sense of Lemma 2.11 (3), if we only assume that M and N both do not have Woodin cardinals, for the following reason.

Let x be a real coding $M \oplus N$. If $M \cap \text{Ord}$ and $N \cap \text{Ord}$ are both not definably Woodin over M and N respectively, then this implies together with the assumption that M and N both do not have Woodin cardinals that M and N both do not have definable Woodin cardinals and we can apply Lemma 2.11. So assume now for example that $M \cap \text{Ord}$ is definably Woodin over M . By the proof of Lemma 2.11 the coiteration of M and N can only fail on the M -side because of the lack of a \mathcal{Q} -structure for an iteration tree \mathcal{T} of limit length on M inside $M_{n-1}^\#(x)|\delta_x$. But in this case

we must have that

$$\mathcal{M}(\mathcal{T}) = \mathcal{M}_\lambda^\mathcal{T},$$

where $\mathcal{M}_\lambda^\mathcal{T}$ is the limit model for \mathcal{T} which exists in V . This implies that the coiteration on the M -side already finished because the M -side can no longer be iterated and by the same argument for N we have that the coiteration is successful, even if $M \cap \text{Ord}$ and $N \cap \text{Ord}$ are both definably Woodin over M and N respectively.

In what follows we also want to consider premice which are not fully ω_1 -iterable but only Π_n^1 -iterable for some $n \in \omega$. This notion was defined by John Steel in [St95] and he proved that for a premouse M the statement “ M is Π_n^1 -iterable” is Π_n^1 -definable uniformly in any code for M . See Definitions 1.4 and 1.6 in [St95] for a precise definition of Π_n^1 -iterability. He proves in Lemma 2.2 in [St95] that for an $(n-1)$ -small premouse N which is ω -sound and such that $\rho_\omega(N) = \omega$, Π_n^1 -iterability is enough to perform the standard comparison arguments with an $(\omega_1 + 1)$ -iterable premouse which has the same properties.

This implies that using Lemma 1.17 and Lemma 2.11 (2) the following version of Lemma 2.2 proven in [St95] holds true for ω_1 -iterability (instead of $(\omega_1 + 1)$ -iterability as in [St95]).

Lemma 2.12. *Let $n \geq 2$ and assume that $M_{n-1}^\#(x)$ exists and is ω_1 -iterable for all reals x . Let M and N be countable premice which have a common cutpoint δ such that M and N are $(n-1)$ -small above δ . Assume that M and N are δ -sound and that $\rho_\omega(M) \leq \delta$ and $\rho_\omega(N) \leq \delta$.*

- (1) *Assume that M is ω_1 -iterable above δ . Let \mathcal{T} be a normal iteration tree on M above δ of length λ for some limit ordinal $\lambda < \omega_1$ and let b be the unique cofinal well-founded branch through \mathcal{T} such that $\mathcal{Q}(b, \mathcal{T})$ is ω_1 -iterable above $\delta(\mathcal{T})$. Then b is the unique cofinal branch c through \mathcal{T} such that $\mathcal{M}_c^\mathcal{T}$ is well-founded and, if $n \geq 3$, $\mathcal{Q}(c, \mathcal{T})$ is Π_{n-1}^1 -iterable above $\delta(\mathcal{T})$.*
- (2) *Assume M is ω_1 -iterable above δ . Then M is Π_n^1 -iterable above δ .*
- (3) *Assume M is ω_1 -iterable above δ and N is Π_n^1 -iterable above δ . Moreover assume that*

$$M|\delta = N|\delta.$$

Then we can successfully coiterate M and N above δ , that means we have that

$$M \trianglelefteq N \text{ or } N \trianglelefteq M.$$

Proof. Apply Lemma 2.2 from [St95] inside the model $M_{n-1}^\#(x)$, where x is a real coding $M \oplus N$. This immediately gives Lemma 2.12 using Lemma 2.11 (2), because we have that

$$\omega_2^{M_{n-1}^\#(x)} < \omega_1^V$$

and therefore M is $(\omega_1 + 1)$ -iterable inside $M_{n-1}^\#(x)$ and by Lemma 1.17 we have that N is Π_n^1 -iterable inside the model $M_{n-1}^\#(x)$ since $M_{n-1}^\#(x)$ is Σ_n^1 -correct in V . \square

Remark. Similarly as for Lemma 2.11 this lemma also holds true in the special case that $\delta = \omega$. More precisely in this case Lemma 2.12 (3) holds true in the following sense. If M and N are ω -sound, $\rho_\omega(M) = \omega$ and $\rho_\omega(N) = \omega$, and if we assume ω_1 -iterability and Π_n^1 -iterability for M and N respectively, then we have that

$$M \trianglelefteq N \text{ or } N \trianglelefteq M.$$

In Section 3 we in fact need the following strengthening of Lemma 2.12 for odd n , which is proved in Lemma 3.3 in [St95]. That it holds for ω_1 -iterability instead of $(\omega_1 + 1)$ -iterability follows by the same argument as the one we gave for Lemma 2.12 above. This lemma only holds for odd n because of the periodicity in the projective hierarchy. For more details see [St95].

Lemma 2.13. *Let $n \geq 1$ be odd and assume that $M_{n-1}^\#(x)$ exists and is ω_1 -iterable for all reals x . Let M and N be countable premice which agree up to a common cutpoint δ such that M and N are δ -sound and such that $\rho_\omega(M) \leq \delta$ and $\rho_\omega(N) \leq \delta$. Assume that M is $(n-1)$ -small above δ and N is not $(n-1)$ -small above δ . Moreover assume that M is Π_n^1 -iterable above δ and that N is ω_1 -iterable above δ . Then we have that*

$$M \trianglelefteq N.$$

For the proof of Lemma 2.3 we need the following variant of Lemma 2.12 which is a straightforward consequence of the proof of Lemma 2.2 in [St95], because the assumption that the premice M and N both do not have definable Woodin cardinals yields that \mathcal{Q} -structures exist in a coiteration of M and N .

We say that an iteration tree \mathcal{U} is a *putative iteration tree* if \mathcal{U} satisfies all properties of an iteration tree, but we allow the last model of \mathcal{U} to be ill-founded, in case \mathcal{U} has a last model.

Lemma 2.14. *Let $n \geq 2$ and assume that $M_{n-1}^\#(x)$ exists and is ω_1 -iterable for all reals x . Let M and N be countable premice which have a common cutpoint δ such that M and N are $(n-1)$ -small above δ and solid above δ . Assume that M and N both do not have definable Woodin cardinals and assume in addition that M is ω_1 -iterable above δ and that N is Π_n^1 -iterable above δ . Moreover assume that*

$$M|\delta = N|\delta.$$

Then we can successfully coiterate M and N above δ in the analogous to Lemma 2.11 (3), that means here that there is an iteration tree \mathcal{T} on M and

a putative iteration tree \mathcal{U} on N of length $\lambda + 1$ for some ordinal λ such that we have

$$\mathcal{M}_\lambda^\mathcal{T} \trianglelefteq \mathcal{M}_\lambda^\mathcal{U} \text{ or } \mathcal{M}_\lambda^\mathcal{U} \trianglelefteq \mathcal{M}_\lambda^\mathcal{T}.$$

So in the first case $\mathcal{M}_\lambda^\mathcal{U}$ need not be fully well-founded, but it is well-founded up to $\mathcal{M}_\lambda^\mathcal{T} \cap \text{Ord}$. In the second case we have that $\mathcal{M}_\lambda^\mathcal{U}$ is fully well-founded and \mathcal{U} is in fact an iteration tree.

Analogous to the remark after the proof of Lemma 2.11 we get that the following strengthening of Lemma 2.14 holds true, where we replace “no definable Woodin cardinals” by “no Woodin cardinals”.

Corollary 2.15. *Let $n \geq 2$ and assume that $M_{n-1}^\#(x)$ exists and is ω_1 -iterable for all reals x . Let M and N be countable premice which have a common cutpoint δ such that M and N are $(n-1)$ -small above δ and solid above δ . Assume that M and N both do not have Woodin cardinals and assume in addition that M is ω_1 -iterable above δ and that N is Π_n^1 -iterable above δ . Moreover assume that*

$$M|\delta = N|\delta.$$

Then we can successfully coiterate M and N above δ in the analogous to Lemma 2.11 (3), that means here that there is an iteration tree \mathcal{T} on M and a putative iteration tree \mathcal{U} on N of length $\lambda + 1$ for some ordinal λ such that we have

$$\mathcal{M}_\lambda^\mathcal{T} \trianglelefteq \mathcal{M}_\lambda^\mathcal{U} \text{ or } \mathcal{M}_\lambda^\mathcal{U} \trianglelefteq \mathcal{M}_\lambda^\mathcal{T}.$$

So we again have that in the first case $\mathcal{M}_\lambda^\mathcal{U}$ need not be fully well-founded, but it is well-founded up to $\mathcal{M}_\lambda^\mathcal{T} \cap \text{Ord}$, and in the second case we have that $\mathcal{M}_\lambda^\mathcal{U}$ is fully well-founded and \mathcal{U} is in fact an iteration tree.

Proof. To simplify the notation we again assume that $\delta = \omega$. Analogous to the remark after the proof of Lemma 2.11 we can just use Lemma 2.14 in the case that $M \cap \text{Ord}$ and $N \cap \text{Ord}$ are both not definably Woodin over M and N respectively, because in this case M and N in fact do not have definable Woodin cardinals.

So assume for example that $N \cap \text{Ord}$ is definably Woodin over N . Let x be a real coding $M \oplus N$ and consider the coiteration of M and N inside $M_{n-1}^\#(x)$. Let \mathcal{T} and \mathcal{U} be the resulting trees on M and N respectively. Assume that the coiteration breaks down on the N -side, that means \mathcal{U} is an iteration tree of limit length λ such that there is no \mathcal{Q} -structure $\mathcal{Q}(\mathcal{U})$ for \mathcal{U} in $M_{n-1}^\#(x)$. Since N has no Woodin cardinals, this can only be the case if

$$\mathcal{M}(\mathcal{U}) = \mathcal{M}_\lambda^\mathcal{U},$$

where $\mathcal{M}_\lambda^\mathcal{U}$ is the limit model for \mathcal{U} which exists in V since N is Π_n^1 -iterable in V . Therefore we have as in the remark after the proof of Lemma 2.11 that the coiteration on the N -side already finished.

If we assume that $M \cap \text{Ord}$ is definably Woodin over M it follows by the same argument that the coiteration on the M -side already finished if it breaks down, because M is ω_1 -iterable in V .

Therefore the coiteration of M and N is successful in the sense of Corollary 2.15 even if $M \cap \text{Ord}$ or $N \cap \text{Ord}$ or both of them are definably Woodin over M or N respectively. \square

We now aim to fix an order on OD-sets. Therefore we first introduce the following notion.

Definition 2.16. *Let $x \in \text{OD}$. Then we say $(n, (\alpha_0, \dots, \alpha_n), \varphi)$ is the minimal triple defining x iff $(n, (\alpha_0, \dots, \alpha_n), \varphi)$ is the minimal triple according to the lexicographical order on triples (using the lexicographical order on tuples of ordinals of length $n + 1$ and the order on Gödel numbers for formulae) such that*

$$x = \{z \mid V_{\alpha_0} \models \varphi(z, \alpha_1, \dots, \alpha_n)\}.$$

Definition 2.17. *Let $x, y \in \text{OD}$. Moreover let $(n, (\alpha_0, \dots, \alpha_n), \varphi)$ and $(m, (\beta_0, \dots, \beta_m), \psi)$ be the minimal triples defining x and y respectively as in Definition 2.16. Then we say x is less than y in the order on the OD-sets and write*

$$x <_{\text{OD}} y,$$

iff $(n, (\alpha_0, \dots, \alpha_n), \varphi)$ is smaller than $(m, (\beta_0, \dots, \beta_m), \psi)$ in the lexicographical order on triples, that means iff either

- (a) $n < m$, or
- (b) $n = m$ and $(\alpha_0, \dots, \alpha_n) <_{\text{lex}} (\beta_0, \dots, \beta_m)$, where $<_{\text{lex}}$ denotes the lexicographical order on tuples of ordinals of length $n + 1$, or
- (c) $(\alpha_0, \dots, \alpha_n) = (\beta_0, \dots, \beta_m)$ and $\ulcorner \varphi \urcorner < \ulcorner \psi \urcorner$, where $\ulcorner \varphi \urcorner$ and $\ulcorner \psi \urcorner$ denote the Gödel numbers of the formulae φ and ψ respectively.

Using this order on OD-sets we can prove the following lemma, which will also be used in the proof of Lemma 2.3.

Lemma 2.18. *Let $n \geq 2$ and assume that $M_{n-1}^\#(z)$ exists and is ω_1 -iterable for all reals z . Moreover let x be a real. Then we have for all reals $y \geq_T x$ such that $y \in M_{n-1}(x)$, that the premice $M_{n-1}(x)$ and $L[E](y)^{M_{n-1}(x)}$ have the same sets of reals and the same OD-sets of reals in the same order.*

Remark. With $L[E](y)^{M_{n-1}(x)}$ we denote the resulting model of a fully back-grounded extender construction above the real y as in [MS94] performed inside the model $M_{n-1}(x)$, but with the smallness hypothesis weakened to allow ω -small premice in the construction. For more details about such a construction in a more general setting see also [St93].

Proof of Lemma 2.18. We start with some general remarks about the premice we consider. The premouse $L[E](y)^{M_{n-1}(x)}$ still has $n - 1$ Woodin cardinals and is ω_1 -iterable via an iteration strategy which is induced by the

ω_1 -iteration strategy for $M_{n-1}(x)$ by §11 and §12 in [MS94]. Therefore the premouse

$$L[E](x)^{L[E](y)^{M_{n-1}(x)}},$$

which again denotes the fully backgrounded extender construction in the sense of [MS94] with the smallness hypothesis weakened to allow not only 1-small premice, but now performed inside the model $L[E](y)^{M_{n-1}(x)}$, also has $n - 1$ Woodin cardinals and is ω_1 -iterable.

Claim 1. *The premice $L[E](y)^{M_{n-1}(x)}$ and $L[E](x)^{L[E](y)^{M_{n-1}(x)}}$ as defined above are $(n - 1)$ -small.*

Proof. We first show that the y -premouse $L[E](y)^{M_{n-1}(x)}$ is $(n - 1)$ -small, using the $(n - 1)$ -smallness of $M_{n-1}(x)$. Then it follows by the same argument that the x -premouse $L[E](x)^{L[E](y)^{M_{n-1}(x)}}$ is $(n - 1)$ -small.

So assume toward a contradiction that the premouse $L[E](y)^{M_{n-1}(x)}$ is not $(n - 1)$ -small and let $N_y^\#$ be the shortest initial segment of $L[E](y)^{M_{n-1}(x)}$ which is not $(n - 1)$ -small. That means we choose $N_y^\# \trianglelefteq L[E](y)^{M_{n-1}(x)}$ such that it is not $(n - 1)$ -small, but every proper initial segment of $N_y^\#$ is $(n - 1)$ -small. In particular $N_y^\#$ is an active y -premouse, so let F be the top extender of $N_y^\#$. Moreover let N_y be the model obtained from $N_y^\#$ by iterating the top extender F out of the universe inside the model $M_{n-1}(x)$. Now consider $L[E](x)^{N_y}$ and let N be the active x -premouse obtained by adding $F \cap L[E](x)^{N_y}$ as a top extender to an initial segment of $L[E](x)^{N_y}$, analogous to Section 2 in [FNS10] to ensure that N is a premouse. The main result in [FNS10] yields that N is ω_1 -iterable in V , because $L[E](y)^{M_{n-1}(x)}$, $N_y^\#$ and $L[E](x)^{N_y}$ inherit the iterability from $M_{n-1}(x)$ as is §11 and §12 in [MS94]. Moreover N is not $(n - 1)$ -small.

Let $N_x^\#$ be the shortest initial segment of N which is not $(n - 1)$ -small. By Lemma 2.11 we can successfully compare the x -premouse $N_x^\#$ and $M_{n-1}^\#(x)$, because both premice are ω_1 -iterable in V and every proper initial segment of one of them is $(n - 1)$ -small. Therefore we have that in fact $N_x^\# = M_{n-1}^\#(x)$ and thus

$$\mathbb{R} \cap N_x^\# = \mathbb{R} \cap M_{n-1}^\#(x) = \mathbb{R} \cap M_{n-1}(x).$$

But $N_x^\#$ is by construction a countable premouse in $M_{n-1}(x)$, so this is a contradiction. \square

Now fix a real z which codes the x -premouse $M_{n-1}^\#(x)$ and work inside the model $M_{n-1}(z)$ for a while.

We have that every proper initial segment of $M_{n-1}^\#(x)$ is $(n - 1)$ -small, $\rho_\omega(M_{n-1}^\#(x)) = \omega$, and $M_{n-1}^\#(x)$ is ω -sound. Since

$$\omega_2^{M_{n-1}^\#(z)} < \omega_1^V,$$

this yields that $M_{n-1}^\#(x)$ is $(\omega_1 + 1)$ -iterable inside $M_{n-1}^\#(z)$ by Lemma 2.11 (2). So in particular, working inside $M_{n-1}(z)$, the x -premouse $M_{n-1}(x)$, obtained from $M_{n-1}^\#(x)$ by iterating the top measure out of the universe, is $(\omega_1 + 1)$ -iterable.

Claim 2. *The x -premouse $M_{n-1}(x)$ and $L[E](x)^{L[E](y)^{M_{n-1}(x)}}$ agree below the least measurable cardinal in $M_{n-1}(x)$.*

Proof. The x -premouse $L[E](x)^{L[E](y)^{M_{n-1}(x)}}$ is $(\omega_1 + 1)$ -iterable inside $M_{n-1}(z)$ via an iteration strategy which is induced by the iteration strategy for $M_{n-1}(x)$. In particular we can successfully compare the x -premouse $M_{n-1}(x)$ and $L[E](x)^{L[E](y)^{M_{n-1}(x)}}$ inside the model $M_{n-1}(z)$ using this iteration strategy and they coiterate to the same premouse. This yields the claim. \square

Now we can finally show the following claim, which will finish the proof of Lemma 2.18.

Claim 3. *$M_{n-1}(x)$ and $L[E](y)^{M_{n-1}(x)}$ have the same sets of reals and the same OD-sets of reals in the same order.*

Proof. Claim 2 implies that if κ denotes the least measurable cardinal in $M_{n-1}(x)$, then we have that

$$V_\kappa^{M_{n-1}(x)} \supseteq V_\kappa^{L[E](y)^{M_{n-1}(x)}} \supseteq V_\kappa^{L[E](x)^{L[E](y)^{M_{n-1}(x)}}} = V_\kappa^{M_{n-1}(x)},$$

and therefore

$$V_\kappa^{M_{n-1}(x)} = V_\kappa^{L[E](y)^{M_{n-1}(x)}}.$$

Thus we can consider $M_{n-1}(x)$ and $L[E](y)^{M_{n-1}(x)}$ as $V_\kappa^{M_{n-1}(x)}$ -premouse and, still working in $M_{n-1}(z)$, we can successfully compare them by the same argument as in the proof of Claim 2, using the iteration strategy for $L[E](y)^{M_{n-1}(x)}$ which is induced by the iteration strategy for $M_{n-1}(x)$. As $V_\kappa^{M_{n-1}(x)}$ -premouse $M_{n-1}(x)$ and $L[E](y)^{M_{n-1}(x)}$ coiterate to the same model and hence they have the same sets of reals and the same OD-sets of reals in the same order. \square

This proves Lemma 2.18. \square

Motivated by this lemma we introduce the following notation.

Definition 2.19. *For premice M and N we write $M \sim N$ iff M and N have the same sets of reals and the same OD-sets of reals in the same order.*

In the proof of Lemma 2.3 we will also need the following lemma.

Lemma 2.20. *Let M be an ω_1 -iterable premouse with a Woodin cardinal and let $\delta = \delta_M$ denote the least Woodin cardinal in M . Let*

$$(\mathcal{M}_\xi, \mathcal{N}_\xi \mid \xi \in \text{Ord})$$

be the sequence of models obtained from a fully backgrounded extender construction inside M in the sense of [MS94] but with the smallness hypothesis weakened, such that

$$\mathcal{M}_{\xi+1} = \mathcal{C}_\omega(\mathcal{N}_{\xi+1})$$

and let $L[E]$ be the resulting model. Then we have for all $\xi \geq \delta$ that

$$\rho_\omega(\mathcal{M}_\xi) \geq \delta,$$

and therefore

$$\mathcal{M}_\delta = L[E]|\delta.$$

Remark. This lemma also holds true if we perform the fully backgrounded extender construction to obtain the model $L[E]$ relativized to a real $x \in M$, which we then denote by $L[E](x)$.

Proof of Lemma 2.20. Work in M . Assume not and let \mathcal{M}_ξ be the least model with $\xi \geq \delta$ such that $\rho = \rho_\omega(\mathcal{M}_\xi) < \delta$.

The background universe $M|\delta$ is generic over \mathcal{M}_ξ via the δ -version of the Extender Algebra \mathbb{Q}_δ (see Lemma 1.3 in [SchSt09]), because all extenders that are part of the fully backgrounded extender construction satisfy the axioms of the Extender Algebra. We have that \mathbb{Q}_δ satisfies the δ -chain condition by the same argument as in the proof that the “classical” version of the Extender Algebra satisfies the δ -chain condition (see Theorem 7.14 in [St10] for the “classical” version and Lemma 1.3 in [SchSt09] for the δ -version of the Extender Algebra).

By definition of ρ there exists an $r\Sigma_{n+1}$ -formula ϕ for some $n < \omega$ and a parameter $b \in \mathcal{M}_\xi$ such that

$$a \stackrel{\text{def}}{=} \{x < \rho \mid \mathcal{M}_\xi \models \phi(x, b)\} \notin \mathcal{M}_\xi.$$

But we have that $a \in \mathcal{M}_\xi[M|\delta]$ because $a \subseteq \rho < \delta$ and we are working inside the model M . Moreover the set a is bounded in δ , so there exists a name $\tau \in \mathcal{M}_\delta$ such that $\tau^{M|\delta} = a$. Then there exists a condition $p \in \mathbb{Q}_\delta$ such that

$$p \Vdash_{\mathbb{Q}_\delta}^{\mathcal{M}_\xi} \text{“}\tau = \{x < \check{\rho} \mid \mathcal{M}_\xi \models \phi(x, \check{b})\}\text{”}.$$

Therefore

$$a = \{x < \rho \mid p \Vdash_{\mathbb{Q}_\delta}^{\mathcal{M}_\xi} \text{“}\check{x} \in \tau\text{”}\}.$$

Note that \mathbb{Q}_δ is definable over \mathcal{M}_δ .

Claim 1. *We have that*

$$p \Vdash_{\mathbb{Q}_\delta}^{\mathcal{M}_\xi} \text{“}\check{x} \in \tau\text{”} \text{ iff } p \Vdash_{\mathbb{Q}_\delta}^{\mathcal{M}_\delta} \text{“}\check{x} \in \tau\text{”}.$$

Proof. Suppose we have that $p \Vdash_{\mathbb{Q}_\delta}^{\mathcal{M}_\xi} \text{“}\check{x} \in \tau\text{”}$ and $p \not\Vdash_{\mathbb{Q}_\delta}^{\mathcal{M}_\delta} \text{“}\check{x} \in \tau\text{”}$. Fix a condition $q \leq p$ such that

$$q \Vdash_{\mathbb{Q}_\delta}^{\mathcal{M}_\delta} \text{“}\neg \check{x} \in \tau\text{”}.$$

Pick an \mathcal{M}_ξ generic filter g such that $q \in g$. This contradicts $p \Vdash_{\mathbb{Q}_\delta}^{\mathcal{M}_\xi} \text{“}\dot{x} \in \tau\text{”}$, because g is also generic over \mathcal{M}_δ .

For the converse suppose that we have $p \Vdash_{\mathbb{Q}_\delta}^{\mathcal{M}_\delta} \text{“}\dot{x} \in \tau\text{”}$ and $p \not\Vdash_{\mathbb{Q}_\delta}^{\mathcal{M}_\xi} \text{“}\dot{x} \in \tau\text{”}$. Fix a $q \leq p$ such that

$$q \Vdash_{\mathbb{Q}_\delta}^{\mathcal{M}_\xi} \text{“}\neg \dot{x} \in \tau\text{”}.$$

As above pick an \mathcal{M}_ξ generic filter g such that $q \in g$. Then g is also generic over \mathcal{M}_δ again and so this contradicts $p \Vdash_{\mathbb{Q}_\delta}^{\mathcal{M}_\delta} \text{“}\dot{x} \in \tau\text{”}$. \square

Therefore we have that

$$a = \{x < \rho \mid p \Vdash_{\mathbb{Q}_\delta}^{\mathcal{M}_\delta} \text{“}\dot{x} \in \tau\text{”}\}.$$

That means the set a is definable over \mathcal{M}_δ and thus $a \in \mathcal{M}_\delta$. Therefore we have in particular that $a \in \mathcal{M}_\xi$, a contradiction. \square

2.3. OD-Determinacy for an Initial Segment of M_{n-1} . Now we can turn to the proof of Lemma 2.3. Recall

Lemma 2.3. *Let $n \geq 1$. Assume that $M_{n-1}^\#(x)$ exists and is ω_1 -iterable for all $x \in {}^\omega\omega$ and that all Σ_{n+1}^1 -definable sets of reals are determined. Then there exists a real y_0 such that for all reals $x \geq_T y_0$,*

$$M_{n-1}(x)|\delta_x \models \text{OD-determinacy},$$

where δ_x denotes the least Woodin cardinal in $M_{n-1}(x)$ if $n > 1$ and δ_x denotes the least x -indiscernible in $M_0(x) = L[x]$ if $n = 1$.

Proof of Lemma 2.3. For $n = 1$ we have that Lemma 2.3 immediately follows from Theorem 3.1 in [KS85]. So assume that $n > 1$ and assume further toward a contradiction that there is no such real y_0 as in the statement of Lemma 2.3.

Then there are cofinally (in the Turing degrees) many $y \in {}^\omega\omega$ such that there exists an $M \trianglelefteq M_{n-1}(y)|\delta_y$ with

$$M \models \neg\text{OD-determinacy}.$$

We want to consider x -premise M for $x \in {}^\omega\omega$ which have the following properties:

- (1) $M \models \text{ZFC}$, M is countable, and we have that the following formula holds true:

for all $k < \omega$ and for all $(x_n \mid n < k)$ and $(M^n \mid n \leq k)$ such that

$$\begin{aligned} & x_n \in M^n \cap {}^\omega\omega, \quad x_{n+1} \geq_T x_n \geq_T x, \\ & M^0 = M, \quad M^{n+1} = L[E](x_n)^{M^n}, \quad \text{and we have} \\ & \text{for all } n < k \text{ that } M^{n+1} \sim M^n \text{ and} \\ & M^n \text{ does not have a Woodin cardinal,} \end{aligned}$$

where the relation “ \sim ” is as defined in Definition 2.19,

(2) M is $(n - 1)$ -small, and

$$M \models \neg\text{OD-determinacy},$$

(3) for all $\xi < M \cap \text{Ord}$ such that $M|\xi$ satisfies property (1),

$$M|\xi \models \text{OD-determinacy},$$

and

(4) M is Π_n^1 -iterable.

If M is an x -premouse, then we have for every real y such that $y \geq_T x$ and $y \in M$, that

$$M \models (1) \Rightarrow L[E](y)^M \models (1),$$

where $L[E](y)^M$ as above denotes the resulting model of a fully backgrounded extender construction above the real y inside M as in [MS94] with the smallness hypothesis weakened as for example in the remark after Lemma 2.18.

We first show that there exists a Turing cone of reals x such that there exists an x -premouse M satisfying properties (1) - (4) as above.

Claim 2. *For cofinally many $x \in {}^\omega\omega$, $M_{n-1}(x)|\delta_x$ satisfies properties (1) and (2).*

Proof. By assumption there are cofinally many $x \in {}^\omega\omega$ such that

$$M_{n-1}(x)|\delta_x \models \neg\text{OD-determinacy}.$$

Pick such an $x \in {}^\omega\omega$. Then $M_{n-1}(x)|\delta_x$ is obviously a countable ZFC model without a Woodin cardinal. Moreover it is a proper initial segment of $M_{n-1}(x)$ and therefore $(n - 1)$ -small. This already implies that property (2) holds true for $M_{n-1}(x)|\delta_x$. Lemma 2.18 yields that

$$M_{n-1}(x) \sim L[E](y)^{M_{n-1}(x)},$$

for all reals $y \geq_T x$ such that $y \in M_{n-1}(x)$. Then we have for such a real y that

$$M_{n-1}(x)|\delta_x \sim L[E](y)^{M_{n-1}(x)}|\delta_x = L[E](y)^{M_{n-1}(x)|\delta_x},$$

where the \sim -equivalence follows from Lemma 2.18 and the equality is given by Lemma 2.20. Therefore the formula in property (1) holds true for $M_{n-1}(x)|\delta_x$ as we also have that $L[E](y)^{M_{n-1}(x)|\delta_x}$ does not have a Woodin cardinal. \square

By Claim 2 there are cofinally many $x \in {}^\omega\omega$ such that there exists an x -premouse M which satisfies properties (1) - (4) defined above. Such x -premise M can be obtained by taking the smallest initial segment of $M_{n-1}(x)|\delta_x$ which satisfies properties (1) - (4) defined above. Moreover the set

$$A = \{x \in {}^\omega\omega \mid \text{there is an } x\text{-premouse } M \text{ with (1) - (4)}\}$$

is Σ_{n+1}^1 -definable and Turing invariant. So by Σ_{n+1}^1 (Turing-)determinacy there exists a cone of such reals $x \in A$, because the set A cannot be completely disjoint from a cone of reals since there are cofinally many reals $x \in A$ as argued above. Let v be a base of this cone and consider a real $x \geq_T v$ in the cone.

Claim 3. *If there is an ω_1 -iterable x -premouse M with properties (1) and (2), then every x -premouse N satisfying properties (1)–(4) is in fact already ω_1 -iterable.*

Proof. Assume there is such an x -premouse M and let N be an arbitrary x -premouse satisfying properties (1)–(4). By Corollary 2.15 we can successfully coiterate the x -premouse M and N because they are both $(n-1)$ -small, solid and without Woodin cardinals and moreover M is ω_1 -iterable and N is Π_n^1 -iterable. Let \mathcal{T} and \mathcal{U} be the resulting trees of length $\lambda + 1$ for some ordinal λ on M and N respectively, in particular \mathcal{U} might be a putative iteration tree.

We consider three possible cases as follows.

Case 1. We have $\mathcal{M}_\lambda^\mathcal{U} \sqsubseteq \mathcal{M}_\lambda^\mathcal{T}$ and the iteration from N to $\mathcal{M}_\lambda^\mathcal{U}$ is without drops on the main branch.

That means we have that $[0, \lambda]_{\mathcal{U}} \cap \mathcal{D}^\mathcal{U} = \emptyset$, where the set $\mathcal{D}^\mathcal{U}$ denotes the set of drops in model or degree in the iteration tree \mathcal{U} (see Section 3.1 in [St10] for a formal definition of $\mathcal{D}^\mathcal{U}$). In this case we have that the model $\mathcal{M}_\lambda^\mathcal{U}$ is fully well-founded and N is elementary embedded into an ω_1 -iterable model and thus ω_1 -iterable itself.

Case 2. We have $\mathcal{M}_\lambda^\mathcal{T} \triangleleft \mathcal{M}_\lambda^\mathcal{U}$ and the iteration from M to $\mathcal{M}_\lambda^\mathcal{T}$ is without drops on the main branch.

In this case $\mathcal{M}_\lambda^\mathcal{U}$ need not be fully well-founded, but this will not affect our argument to follow, because we have that $\mathcal{M}_\lambda^\mathcal{U}$ is well-founded up to $\mathcal{M}_\lambda^\mathcal{T} \cap \text{Ord}$. So there exists an ordinal $\alpha < \mathcal{M}_\lambda^\mathcal{U} \cap \text{Ord}$ such that $\mathcal{M}_\lambda^\mathcal{U}|_\alpha = \mathcal{M}_\lambda^\mathcal{T}$ and we have by elementarity that

$$\mathcal{M}_\lambda^\mathcal{T} \models \neg\text{OD-determinacy},$$

since property (2) holds for M . Since $\alpha = \mathcal{M}_\lambda^\mathcal{T} \cap \text{Ord}$ it follows that

$$\mathcal{M}_\lambda^\mathcal{U}|_\alpha \models \neg\text{OD-determinacy}.$$

Moreover we have that $\mathcal{M}_\lambda^\mathcal{U}|_\alpha \models (1)$, because we have by elementarity that $\mathcal{M}_\lambda^\mathcal{T} \models (1)$. If there is no drop in the iteration from N to $\mathcal{M}_\lambda^\mathcal{U}$, this contradicts the minimality property (3) for N by elementarity. But even if there is a drop on the main branch in \mathcal{U} this statement is transferred along the branch to N by the following argument.

Assume there is a drop at stage $\beta + 1$ on the main branch through \mathcal{U} , that means $\mathcal{M}_{\beta+1}^*$ is a proper initial segment of \mathcal{M}_γ for $\gamma = \text{pred}_{\mathcal{U}}(\beta + 1)$, where $\mathcal{M}_{\beta+1}^*$ is the model to which the next extender F_β from the \mathcal{M}_β sequence is applied in the iteration as introduced in Section 3.1 in [St10]. Since there can only be finitely many drops along the main branch through the iteration tree \mathcal{U} , we can assume further without loss of generality that this is the only drop along the main branch through \mathcal{U} . (If there is more than one drop on the main branch through \mathcal{U} , we repeat the argument to follow for the remaining drops.) Then by elementarity there is an ordinal $\alpha' < \mathcal{M}_{\beta+1}^* \cap \text{Ord} < \mathcal{M}_\gamma \cap \text{Ord}$ such that

$$\mathcal{M}_{\beta+1}^* \upharpoonright \alpha' \models (1) + \neg\text{OD-determinacy}.$$

But then also

$$\mathcal{M}_\gamma \upharpoonright \alpha' \models (1) + \neg\text{OD-determinacy},$$

and therefore by elementarity there is an ordinal $\alpha'' < N \cap \text{Ord}$ such that

$$N \upharpoonright \alpha'' \models (1) + \neg\text{OD-determinacy}.$$

This now again contradicts the minimality property (3) for N .

Case 3. We have $\mathcal{M}_\lambda^T = \mathcal{M}_\lambda^U$, there is no drop on the main branch in the iteration from M to \mathcal{M}_λ^T , but there is a drop on the main branch in the iteration from N to \mathcal{M}_λ^U .

This immediately is a contradiction because it implies that we have $\mathcal{M}_\lambda^T \models \text{ZFC}$, but at the same time $\rho_\omega(\mathcal{M}_\lambda^U) < \mathcal{M}_\lambda^U \cap \text{Ord}$. \square

Since by Claim 2 there are cofinally many reals x such that the ω_1 -iterable x -premouse $M_{n-1}(x) \upharpoonright \delta_x$ satisfies properties (1) and (2), Claim 3 yields that the following claim holds true.

Claim 4. *There are cofinally many reals x such that every x -premouse satisfying properties (1) – (4) is in fact ω_1 -iterable.*

Consider the game G which is defined as follows.

$$\begin{array}{c|c} \text{I} & x \oplus a \\ \hline \text{II} & y \oplus b \end{array} \quad \text{for } x, a, y, b \in {}^\omega\omega.$$

The players I and II alternate playing natural numbers and the game lasts ω steps. Say player I produces a real $x \oplus a$ and player II produces a real $y \oplus b$. Then player I wins G iff there exists an $(x \oplus y)$ -premouse M which satisfies properties (1) – (4) and if A_M denotes the least OD-set of reals in M which is not determined, then $a \oplus b \in A_M$.

This game is Σ_{n+1}^1 -definable and therefore determined. So say first player I has a winning strategy τ for G . Recall that v denotes a base for a cone of reals x such that there exists an x -premouse which satisfies properties

(1) – (4) and pick a real $z^* \geq_T x_\tau \oplus v$, where x_τ is a real coding the winning strategy τ for player I, such that every z^* -premouse which satisfies properties (1) – (4) is in fact ω_1 -iterable (using Claim 4).

We now aim to construct a real $z \geq_T z^*$ such that there is a z -premouse N which satisfies properties (1) – (4), is ω_1 -iterable, and satisfies the following additional property (3*).

(3*) For all reals $y \in N$ such that $y \geq_T z$ there exists no ordinal $\xi < N \cap \text{Ord}$ such that

$$L[E](y)^N \upharpoonright \xi \models (1) + \neg \text{OD-determinacy}.$$

Let M^0 be an arbitrary z^* -premouse satisfying properties (1) – (4), which is therefore ω_1 -iterable. Assume that if we let $N = M^0$ and $z = z^*$, then property (3*) is not satisfied. So there is a real $y \in M^0$ with $y \geq_T z^*$ witnessing the failure of property (3*) in M^0 . Let $y_0 \in M^0$ with $y_0 \geq_T z^*$ be a witness for that fact such that the ordinal $\xi^0 < L[E](y_0)^{M^0} \cap \text{Ord}$ with

$$L[E](y_0)^{M^0} \upharpoonright \xi^0 \models (1) + \neg \text{OD-determinacy}$$

is minimal. Let $M^1 = L[E](y_0)^{M^0} \upharpoonright \xi^0$ and assume that

$$M^1 \not\models (3^*),$$

because otherwise we could pick $N = M^1$ and $z = y_0$ and the construction would be finished. Then as before there is a real $y_1 \in M^1$ with $y_1 \geq_T y_0$ and a minimal ordinal $\xi^1 < L[E](y_1)^{M^1} \cap \text{Ord} = M^1 \cap \text{Ord} = \xi^0$ such that

$$L[E](y_1)^{M^1} \upharpoonright \xi^1 \models (1) + \neg \text{OD-determinacy}.$$

This construction has to stop at a finite stage, because otherwise we have that $\xi^0 > \xi^1 > \dots$ is an infinite descending chain of ordinals. Therefore there is a natural number $n < \omega$ such that

$$M^n = L[E](y_{n-1})^{M^{n-1}} \upharpoonright \xi^{n-1},$$

and

$$M^n \models (3^*).$$

Let $z = y_{n-1}$ and $N = M^n$. Then we have that $z \geq_T z^*$ and N is a z -premouse which satisfies properties (1) and (3*). Moreover by minimality of the ordinal ξ^{n-1} we have that N satisfies property (3). From the construction we also get that

$$N \models \neg \text{OD-determinacy}.$$

Furthermore N inherits properties (2) and (4) and the ω_1 -iterability from M^0 since it is obtained by performing multiple fully backgrounded extender constructions inside the ω_1 -iterable premouse M^0 . Here the fact that N is $(n-1)$ -small follows from the $(n-1)$ -smallness of M^0 by an argument we already gave earlier in the proof of Lemma 2.18. Thus N and z are as desired.

Let A_N denote the least non-determined OD-set of reals in N . We define a strategy τ^* for player I in the usual Gale-Stewart game $G(A_N)$ with payoff set A_N played inside the model N as follows. Assume player II produces the real $b \in N$. Then we consider the following run of the original game G defined above:

$$\frac{\text{I} \mid x \oplus a = \tau((z \oplus b) \oplus b)}{\text{II} \mid (z \oplus b) \oplus b}$$

Player II plays the real $(z \oplus b) \oplus b$ and player I responds with the real $x \oplus a$ according to his winning strategy τ in G . Note that this run of the game G is in the model N . We define the strategy τ^* such that in a run of the game $G(A_N)$ inside N according to τ^* player I has to respond to the real b with producing the real a .

$$\frac{\text{I} \mid a = \tau^*(b)}{\text{II} \mid b}$$

Claim 5. τ^* is a winning strategy for player I in the Gale-Stewart game with payoff set A_N played in N .

This claim implies that A_N is determined in N , contradicting that A_N was assumed to be the least non-determined OD-set of reals in N .

Proof of Claim 5. Since τ is a winning strategy for player I in the original game G , there exists an $(x \oplus (z \oplus b))$ -premouse N' which satisfies properties (1) – (4) such that

$$a \oplus b \in A_{N'},$$

where $A_{N'}$ denotes the least non-determined OD-set of reals in N' .

We want to show that

$$A_{N'} = A_N$$

in order to conclude that τ^* is a winning strategy for player I in the Gale-Stewart game with payoff set A_N played in N .

Property (1) yields that

$$L[E](x \oplus (z \oplus b))^N \sim N,$$

because $x, z, b \in N$ and $x \oplus (z \oplus b) \geq_T z$. Therefore $L[E](x \oplus (z \oplus b))^N$ is an $(x \oplus (z \oplus b))$ -premouse which satisfies property (2), because it has the same sets of reals and the same OD-sets of reals as N and hence

$$L[E](x \oplus (z \oplus b))^N \models \text{-OD-determinacy}.$$

The $(n-1)$ -smallness of $L[E](x \oplus (z \oplus b))^N$ follows from the $(n-1)$ -smallness of N by an argument we already gave earlier in the proof of Lemma 2.18. Moreover $L[E](x \oplus (z \oplus b))^N$ inherits property (1) from N .

Since $L[E](x \oplus (z \oplus b))^N$ is the result of a fully backgrounded extender construction inside the ω_1 -iterable premouse N , it is ω_1 -iterable itself. Therefore Claim 3 yields that in particular the $(x \oplus (z \oplus b))$ -premouse N' is also ω_1 -iterable, because it was chosen such that it satisfies properties (1) – (4). So we can coiterate $L[E](x \oplus (z \oplus b))^N$ and N' by the remark after Lemma 2.11 since they are both $(n-1)$ -small ω_1 -iterable $(x \oplus (z \oplus b))$ -premise which do not have Woodin cardinals. Thus by the minimality of N' from property (3) and an argument analogous to the one we already gave in the proof of Claim 3, we have that

$$L[E](x \oplus (z \oplus b))^N \geq^* N',$$

where \leq^* denotes the usual mouse order¹. Moreover we have minimality for the premouse $L[E](x \oplus (z \oplus b))^N$ in the sense of property (3*) for N . This yields again by an argument analogous to the one we already gave in the proof of Claim 3 that

$$L[E](x \oplus (z \oplus b))^N \leq^* N'.$$

Therefore we have that in fact

$$L[E](x \oplus (z \oplus b))^N =^* N',$$

and hence

$$L[E](x \oplus (z \oplus b))^N \sim N'.$$

Using $L[E](x \oplus (z \oplus b))^N \sim N$ it follows that

$$N \sim N',$$

and thus $A_N = A_{N'}$. □

Now suppose player II has a winning strategy σ in the game G introduced above and recall that v is a base of a cone of reals x such that there exists an x -premouse which satisfies properties (1) – (4). Analogous to the situation when player I has a winning strategy, we pick a real $z^* \geq_T x_\sigma \oplus v$, where x_σ is a real coding the winning strategy σ for player II, such that every z^* -premouse which satisfies properties (1) – (4) is already ω_1 -iterable (see Claims 2 and 3).

As in the argument for player I we can construct a real $z \geq_T z^*$ such that there exists a z -premouse N which satisfies properties (1)–(4), is ω_1 -iterable, and satisfies the additional property (3*). As before we let A_N denote the least non-determined OD-set in N , which exists because the z -premouse N satisfies property (2).

Now we define a strategy σ^* for player II in the usual Gale-Stewart game $G(A_N)$ with payoff set A_N played inside N as follows. Assume that player

¹We say an ω_1 -iterable premouse M is smaller or equal in the *mouse order* than an ω_1 -iterable premouse N and write “ $M \leq^* N$ ” iff M and N successfully coiterate to premice M^* and N^* such that $M^* \trianglelefteq N^*$. Moreover we say that N and M are equal in the mouse order and write “ $M =^* N$ ” iff $M \leq^* N$ and $N \leq^* M$.

I produces a real $a \in N$ in a run of the game $G(A_N)$ inside the model N . Then we consider the following run of the game G :

$$\frac{\text{I} \mid (z \oplus a) \oplus a}{\text{II} \mid y \oplus b = \sigma((z \oplus a) \oplus a)}$$

Player I plays a real $(z \oplus a) \oplus a$ and player II responds with a real $y \oplus b$ according to his winning strategy σ in G . We define the strategy σ^* such that in a run of the game $G(A_N)$ inside the model N according to σ^* player II has to respond to the real a with producing the real b .

$$\frac{\text{I} \mid a}{\text{II} \mid b = \sigma^*(a)}$$

Claim 6. σ^* is a winning strategy for player II in the Gale-Stewart game with payoff set A_N played in N .

This claim implies that A_N is determined in N , again contradicting that A_N was assumed to be the least non-determined OD-set of reals in N .

Proof of Claim 6. We first want to show that the $((z \oplus a) \oplus y)$ -premouse $L[E]((z \oplus a) \oplus y)^N$ satisfies properties (1) – (4).

First property (1) for N yields that

$$L[E]((z \oplus a) \oplus y)^N \sim N,$$

because we have $z, a, y \in N$. Therefore $L[E]((z \oplus a) \oplus y)^N$ is a $((z \oplus a) \oplus y)$ -premouse which satisfies property (2), because it has the same sets of reals and the same OD-sets of reals as N and hence as before

$$L[E]((z \oplus a) \oplus y)^N \models \text{--OD-determinacy}.$$

The $(n - 1)$ -smallness of $L[E]((z \oplus a) \oplus y)^N$ again follows by an argument we already gave in the proof of Lemma 2.18. Moreover $L[E]((z \oplus a) \oplus y)^N$ inherits condition (1) from N .

Since $L[E]((z \oplus a) \oplus y)^N$ is a fully backgrounded extender construction inside the ω_1 -iterable mouse N it is ω_1 -iterable itself by §12 in [MS94]. Therefore $L[E]((z \oplus a) \oplus y)^N$ satisfies properties (1), (2) and (4). By property (3*) for the z -premouse N we additionally have that $L[E]((z \oplus a) \oplus y)^N$ also satisfies property (3).

Since σ is a winning strategy for player II in the original game G , we have that if a, b, z and y are as above, then

$$a \oplus b \notin A_{N'},$$

for all $((z \oplus a) \oplus y)$ -premouse N' satisfying properties (1) – (4), where $A_{N'}$ denotes the least non-determined OD-set of reals in N' . Now let

$$N' = L[E]((z \oplus a) \oplus y)^N,$$

so we have that in particular N' is a $((z \oplus a) \oplus y)$ -premouse and satisfies properties (1) – (4) as above.

As in the previous case where we assumed that player I has a winning strategy in G , we want to show that

$$A_{N'} = A_N$$

in order to conclude that σ^* is a winning strategy for player II in the Gale-Stewart game with payoff set A_N played inside N , using that N' satisfies properties (1) – (4) as argued above.

Using $L[E]((z \oplus a) \oplus y)^N \sim N$ it follows that

$$N \sim N',$$

and thus $A_N = A_{N'}$, as desired. \square

This finishes the proof of Lemma 2.3. \square

2.4. Applications. This section is devoted to two important corollaries of Lemma 2.3 which are going to be used in Sections 2.5 and 3.6.

Corollary 2.21. *Let $n \geq 1$. Assume that $M_{n-1}^\#(x)$ exists and is ω_1 -iterable for all $x \in {}^\omega\omega$ and that all Σ_{n+1}^1 -definable sets of reals are determined. Then*

$$\omega_1^{M_{n-1}(x)} \text{ is measurable in } \text{HOD}^{M_{n-1}(x)|\delta_x},$$

for a cone of reals x .

Proof. This follows from Lemma 2.3 with a generalized version of Solovay's theorem that, under the Axiom of Determinacy AD, ω_1 is measurable. For the readers convenience, we will present a proof of this result, following the proof of the classical result as in Theorem 12.18 (b) in [Sch14] or Lemma 6.2.2 in [SchSt].

Lemma 2.3 yields that

$$M_{n-1}(x)|\delta_x \models \text{OD-determinacy},$$

for a cone of reals x . Let $x \in {}^\omega\omega$ be an arbitrary element of this cone and let us work inside $M_{n-1}(x)|\delta_x$ for the rest of the proof. We aim to define a $< \omega_1$ -complete ultrafilter \mathcal{U} inside $\text{HOD}^{M_{n-1}(x)|\delta_x}$ on $\omega_1 \stackrel{\text{def}}{=} \omega_1^{M_{n-1}(x)|\delta_x} = \omega_1^{M_{n-1}(x)}$, witnessing that ω_1 is measurable in $\text{HOD}^{M_{n-1}(x)|\delta_x}$.

Let $n, m \mapsto \langle n, m \rangle$ be the Gödel pairing function for $n, m < \omega$ and recall that

$$\text{WO} \stackrel{\text{def}}{=} \{x \in {}^\omega\omega \mid R_x \text{ is a well-ordering}\},$$

where we let $(n, m) \in R_x$ iff $x(\langle n, m \rangle) = 1$ for $x \in {}^\omega\omega$. For $y \in \text{WO}$ we write $\|y\|$ for the order type of R_y and for $x \in {}^\omega\omega$ we let

$$|x| = \sup\{\|y\| \mid y \in \text{WO} \wedge y \equiv_T x\}.$$

Consider the set $S = \{|x| \mid x \in {}^\omega\omega\}$ and let $\pi : \omega_1 \rightarrow S$ be an order isomorphism. Now we define the filter \mathcal{U} on ω_1 as follows. For $A \subset \omega_1$ such that $A \in \text{OD}$ we let

$$A \in \mathcal{U} \text{ iff } \{x \in {}^\omega\omega \mid |x| \in \pi''A\} \text{ contains a cone of reals.}$$

Claim 1. $\mathcal{U} \cap \text{HOD}$ is a $< \omega_1^{M_{n-1}(x)}$ -complete ultrafilter in HOD.

Proof. The set $\{x \in {}^\omega\omega \mid |x| \in \pi''A\}$ is Turing invariant. Therefore we have that OD-Turing-determinacy implies that the set $\{x \in {}^\omega\omega \mid |x| \in \pi''A\}$ for $A \in \text{OD}$ either contains a cone of reals or is completely disjoint from a cone of reals. Hence we have that $\mathcal{U} \cap \text{HOD}$ is an ultrafilter on $\omega_1 = \omega_1^{M_{n-1}(x)}$ in HOD. Moreover the following argument shows that this ultrafilter $\mathcal{U} \cap \text{HOD}$ is $< \omega_1^{M_{n-1}(x)}$ -complete in HOD.

Let $\{A_\alpha \mid \alpha < \gamma\} \subset \mathcal{U} \cap \text{HOD}$ be such that $\{A_\alpha \mid \alpha < \gamma\} \in \text{HOD}$ for an ordinal $\gamma < \omega_1^{M_{n-1}(x)}$. Then there is a sequence $(a_\alpha \mid \alpha < \gamma)$ of reals such that for each $\alpha < \gamma$, the real a_α is a base for a cone of reals contained in $\{x \in {}^\omega\omega \mid |x| \in \pi''A_\alpha\}$. Since $\gamma < \omega_1^{M_{n-1}(x)}$, we can fix a bijection $f : \omega \rightarrow \gamma$ in $M_{n-1}(x)$. But then $\bigoplus_{n < \omega} a_{f(n)}$ is a base for a cone of reals contained in

$$\bigcap_{\alpha < \gamma} \{x \in {}^\omega\omega \mid |x| \in \pi''A_\alpha\} = \{x \in {}^\omega\omega \mid |x| \in \pi'' \bigcap_{\alpha < \gamma} A_\alpha\}.$$

So we have that $\bigcap_{\alpha < \gamma} A_\alpha \in \mathcal{U} \cap \text{HOD}$ and thus the filter $\mathcal{U} \cap \text{HOD}$ is $< \omega_1^{M_{n-1}(x)}$ -complete. \square

Therefore $\mathcal{U} \cap \text{HOD}$ witnesses that $\omega_1^{M_{n-1}(x)}$ is measurable in $\text{HOD}^{M_{n-1}(x)|\delta_x}$. \square

In what follows we will prove that in the same situation as above $\omega_2^{M_{n-1}(x)}$ is strongly inaccessible in $\text{HOD}^{M_{n-1}(x)|\delta_x}$, which is another consequence of Lemma 2.3. This is going to be used later in Section 3.6.

In fact the following theorem holds true. It is due to the third author and a consequence of the ‘‘Generation Theorems’’ in [KW10] (see Theorem 5.4 in [KW10]).

Theorem 2.22. *Let $n \geq 1$. Assume that $M_{n-1}^\#(x)$ exists and is ω_1 -iterable for all $x \in {}^\omega\omega$ and that all Σ_{n+1}^1 -definable sets of reals are determined. Then for a cone of reals x ,*

$$\omega_2^{M_{n-1}(x)} \text{ is a Woodin cardinal in } \text{HOD}^{M_{n-1}(x)|\delta_x}.$$

In order to make this paper more self-contained, we shall not use Theorem 2.22 here, though, and we will give a proof of the following theorem, which is essentially due to Moschovakis, and will be used in Section 3.6. A version of it can also be found in [KW10] (see Theorem 3.9 in [KW10]).

Theorem 2.23. *Let $n \geq 1$. Assume that $M_{n-1}^\#(x)$ exists and is ω_1 -iterable for all $x \in {}^\omega\omega$ and that all Σ_{n+1}^1 -definable sets of reals are determined. Then for a cone of reals x ,*

$$\omega_2^{M_{n-1}(x)} \text{ is strongly inaccessible in } \text{HOD}^{M_{n-1}(x)|\delta_x}.$$

Proof. Using Lemma 2.3 we have as above that there is a cone of reals x such that

$$M_{n-1}(x)|\delta_x \models \text{OD-determinacy}.$$

Let x be an element of that cone.

Claim 1. *We have that $\omega_2^{M_{n-1}(x)} = (\Theta_0)^{M_{n-1}(x)}$, where*

$$\Theta_0 = \sup\{\alpha \mid \text{there exists an OD-surjection } f : {}^\omega\omega \rightarrow \alpha\}.$$

Proof. Work inside the model $M_{n-1}(x)$. Since CH holds in $M_{n-1}(x)$, it follows that $\Theta_0 \leq \omega_2$. For the other inequality let $\alpha < \omega_2$ be arbitrary. Then there exists an OD_x -surjection $g : {}^\omega\omega \rightarrow \alpha$ because by definability (using the definability results from [St95]) we have that

$$M_{n-1}(x)|\omega_2^{M_{n-1}(x)} \subseteq \text{HOD}_x^{M_{n-1}(x)}.$$

This implies that there is an OD-surjection $f : {}^\omega\omega \times {}^\omega\omega \rightarrow \alpha$ by varying x and thus it follows that $\alpha \leq \Theta_0$. \square

Work inside the model $M_{n-1}(x)|\delta_x$ from now on and note that it is trivial that $\Theta_0 = \omega_2$ is a regular cardinal in HOD. So we focus on proving that ω_2 is a strong limit. For this purpose we fix an arbitrary ordinal $\alpha < \omega_2$ and prove that $|\mathcal{P}(\alpha)^{\text{HOD}}| < \omega_2$.

Since $\alpha < \omega_2 = \Theta_0$, we can fix a surjection $f : {}^\omega\omega \rightarrow \alpha$ such that $f \in \text{OD}$. This surjection f induces a prewellordering $\leq_f \in \text{OD}$ on ${}^\omega\omega$ if we let

$$x \leq_f y \text{ iff } f(x) \leq f(y)$$

for $x, y \in {}^\omega\omega$. Now consider the pointclass $\Sigma_1^1(\leq_f)$ which is defined as follows. For a set of reals A (or analogously for a set $A \subset ({}^\omega\omega)^k$ for some $k < \omega$) we say $A \in \Sigma_1^1(\leq_f)$ iff there is a Σ_0 -formula φ and a real $z \in {}^\omega\omega$ such that

$$A = \{y \in {}^\omega\omega \mid \exists x \in {}^\omega\omega \varphi(y, x, \leq_f, {}^\omega\omega \setminus \leq_f, z)\}.$$

The pointclass $\Sigma_1^1(\leq_f)$ is defined analogous without the parameter z . We have that there exists a universal $\Sigma_1^1(\leq_f)$ -definable set $U \subseteq {}^\omega\omega \times {}^\omega\omega$ for the pointclass $\Sigma_1^1(\leq_f)$. So we have that for every $\Sigma_1^1(\leq_f)$ -definable set A there exists a $z \in {}^\omega\omega$ such that $A = U_z = \{x \in {}^\omega\omega \mid (z, x) \in U\}$. Now it suffices to prove the following claim.

Claim 2. *Let $X \subset \alpha$ with $X \in \text{OD}$ be arbitrary. Then there exists a $\Sigma_1^1(\leq_f)$ -definable set $A \subset {}^\omega\omega$ such that $X = f''A$.*

Using Claim 2 we can define a surjection

$$g : {}^\omega\omega \rightarrow \mathcal{P}(\alpha) \cap \text{OD}$$

such that $g \in \text{OD}$ by letting $g(z) = f''U_z$ for $z \in {}^\omega\omega$. This yields that we have $|\mathcal{P}(\alpha)^{\text{HOD}}| < \omega_2$ as desired.

Therefore we are left with proving Claim 2 to finish the proof of Theorem 2.23. The proof of Claim 2 is mainly a special case of Moschovakis' Coding Lemma as in Theorem 3.2 in [KW10], so we will outline the proof in this special case.

Proof of Claim 2. Let $X \in \mathcal{P}(\alpha) \cap \text{OD}$ be arbitrary. We aim to show that there is a real $z \in {}^\omega\omega$ such that $X = f''U_z$. Let

$$B = \{z \in {}^\omega\omega \mid f''U_z \subseteq X\}.$$

Moreover let α_z for $z \in B$ be the minimal ordinal β such that $\beta \in X \setminus f''U_z$, if it exists. We aim to show that there exists a real $z \in B$ such that α_z does not exist. So assume toward a contradiction that the ordinal α_z exists for all $z \in B$.

Now consider the following game G of length ω , where player I and player II alternate playing natural numbers such that in the end player I plays a real x and player II plays a real y .

$$\frac{\text{I}}{\text{II}} \left| \begin{array}{c} x \\ y \end{array} \right. \text{ for } x, y \in {}^\omega\omega.$$

We define that player I wins the game G iff

$$x \in B \wedge (y \in B \rightarrow \alpha_x \geq \alpha_y).$$

Note that we have $B \in \text{OD}$ since $f, U, X \in \text{OD}$. Therefore the game G is OD and thus determined by our hypothesis.

Assume first that player I has a winning strategy σ in G . For a real y let $(\sigma * y)_I$ denote player I's moves in a run of the game G , where player II plays y and player I responds according to his winning strategy σ . Then there exists a real z_0 such that

$$U_{z_0} = \bigcup \{U_{(\sigma * y)_I} \mid y \in {}^\omega\omega\}$$

because the right hand side of this equation is $\Sigma_1^1(\leq_f)$ -definable by choice of U . Since σ is a winning strategy for player I, we have that $(\sigma * y)_I \in B$ for all $y \in {}^\omega\omega$ and thus it follows that $z_0 \in B$. Moreover we have that $\alpha_{(\sigma * y)_I} \leq \alpha_{z_0}$ for all $y \in {}^\omega\omega$ by definition of z_0 .

Now we aim to construct a play z^* for player II defeating the strategy σ . Since $f : {}^\omega\omega \rightarrow \alpha$ is a surjection we can choose $a \in {}^\omega\omega$ such that $f(a) = \alpha_{z_0}$. Moreover we let $z^* \in {}^\omega\omega$ be such that $U_{z^*} = U_{z_0} \cup \{a\}$. Then we have that

$$f''U_{z^*} = f''U_{z_0} \cup \{f(a)\} = f''U_{z_0} \cup \{\alpha_{z_0}\} \subset X,$$

since $z_0 \in B$. Hence $z^* \in B$. Moreover we have that

$$\alpha_{z^*} > \alpha_{z_0} \geq \alpha_{(\sigma*y)_I}$$

for all $y \in {}^\omega\omega$. Therefore player II can defeat σ by playing the real z^* , contradicting the fact that σ is a winning strategy for player I.

Assume now that player II has a winning strategy τ in the game G . Let

$$h_0 : {}^\omega\omega \times {}^\omega\omega \rightarrow {}^\omega\omega$$

be a $\Sigma_1^1(\leq_f)$ -definable function such that for all $y, z \in {}^\omega\omega$,

$$U_{h_0(z,y)} = U_z \cap \{x \in {}^\omega\omega \mid f(x) < f(y)\}.$$

Choose $h_1 : {}^\omega\omega \rightarrow {}^\omega\omega$ such that h_1 is $\Sigma_1^1(\leq_f)$ -definable and

$$U_{h_1(z)} = \bigcup \{U_{(h_0(z,y)*\tau)_{II}} \cap \{x \in {}^\omega\omega \mid f(x) = f(y)\} \mid y \in {}^\omega\omega\},$$

where the notion $(h_0(z,y)*\tau)_{II}$ is defined analogous to the corresponding notion for player I introduced above. By Kleene's Recursion Theorem (see for example Theorem 3.1 in [KW10]) there exists a fixed point for h_1 with respect to the set U , that means there exists a real $z^* \in {}^\omega\omega$ such that we have

$$U_{z^*} = U_{h_1(z^*)}.$$

Now our first step is to prove that $z^* \in B$. Assume toward a contradiction that $(f''U_{z^*}) \setminus X \neq \emptyset$ and let $\gamma_0 \in (f''U_{z^*}) \setminus X$ be minimal. Moreover let $y_0 \in U_{z^*}$ be such that $f(y_0) = \gamma_0$. Then

$$\gamma_0 \in f''U_{z^*} = f''U_{h_1(z^*)}$$

and by definition of the function h_1 it follows that $\gamma_0 \in f''U_{(h_0(z^*,y_0)*\tau)_{II}}$. Since γ_0 was picked to be minimal in $(f''U_{z^*}) \setminus X$, we have $h_0(z^*, y_0) \in B$ because we have by definition that

$$U_{h_0(z^*,y_0)} = U_{z^*} \cap \{x \in {}^\omega\omega \mid f(x) < f(y_0)\} = U_{z^*} \cap \{x \in {}^\omega\omega \mid f(x) < \gamma_0\}$$

and thus $f''U_{h_0(z^*,y_0)} \subseteq X$. Since τ is a winning strategy for player II, we have that $(h_0(z^*, y_0)*\tau)_{II} \in B$. Taken all together it follows that

$$\gamma_0 \in f''U_{(h_0(z^*,y_0)*\tau)_{II}} \subseteq X.$$

This contradicts the fact that $\gamma_0 \in (f''U_{z^*}) \setminus X$.

Recall that we assumed toward a contradiction that the ordinal α_{z^*} exists. Let $a^* \in {}^\omega\omega$ be such that

$$f(a^*) = \alpha_{z^*}$$

and note that such an a^* exists since $f : {}^\omega\omega \rightarrow \alpha$ is a surjection and $\alpha_{z^*} < \alpha$. Then we have by definition of the function h_0 that $h_0(z^*, a^*) \in B$ because $z^* \in B$. Moreover we have that $\alpha_{z^*} = \alpha_{h_0(z^*, a^*)}$ holds by definition of α_{z^*} since $f(a^*) = \alpha_{z^*}$. As τ is a winning strategy for player II in the game G , we finally have that

$$\alpha_{(h_0(z^*, a^*)*\tau)_{II}} > \alpha_{h_0(z^*, a^*)} = \alpha_{z^*},$$

because $h_0(z^*, a^*) \in B$. This contradicts the fact that

$$U_{(h_0(z^*, a^*) * \tau)_{\text{II}}} \subset U_{h_1(z^*)} = U_{z^*},$$

by definition of α_{z^*} and $\alpha_{(h_0(z^*, a^*) * \tau)_{\text{II}}}$. Therefore the ordinal α_{z^*} does not exist and thus we finally have that $f''U_{z^*} = X$, as desired. \square

This finishes the proof of Theorem 2.23. \square

2.5. A Proper Class Inner Model with n Woodin Cardinals. In this section we are now able to apply the results from the previous sections to show the existence of a proper class inner model with n Woodin cardinals from determinacy for Π_n^1 - and Π_{n+1}^1 -definable sets (if we assume inductively that Π_n^1 determinacy implies that $M_{n-1}^\#(x)$ exists and is ω_1 -iterable for all $x \in {}^\omega\omega$). This is done in the following theorem, which is a generalization of Theorem 7.7 in [St96] using Lemma 2.3 and Corollary 2.21.

Theorem 2.24. *Let $n \geq 1$. If $M_{n-1}^\#(x)$ exists and is ω_1 -iterable for all $x \in {}^\omega\omega$ and all Σ_{n+1}^1 -definable sets of reals are determined, then there exists a proper class inner model with n Woodin cardinals.*

We are not claiming here that the model obtained in Theorem 2.24 is iterable in any sense. We will show how to construct an ω_1 -iterable premouse with n Woodin cardinals using this model in the next section, but for that we need to assume slightly more determinacy (namely a consequence of determinacy for all Σ_{n+1}^1 -definable sets of reals).

Proof. As before let δ_x denote the least Woodin cardinal in $M_{n-1}(x)$ if $n > 1$ and let δ_x denote the least x -indiscernible in $L[x] = M_0(x)$ if $n = 1$. Then we have that according to Lemma 2.3, there is a real x such that for all reals $y \geq_T x$,

$$M_{n-1}(y)|\delta_y \models \text{OD-determinacy}.$$

Fix such a real x .

In the case $n = 1$ we have that Theorem 2.24 immediately follows from Theorem 7.7 in [St96], so assume $n > 1$.

Let $(K^c)^{M_{n-1}(x)|\delta_x}$ denote the result of a K^c -construction in the sense of Chapter 1 in [St96] performed inside the model $M_{n-1}(x)|\delta_x$. Then we distinguish three cases as follows.

Case 1. Assume that $(K^c)^{M_{n-1}(x)|\delta_x}$ has no Woodin cardinals and is fully iterable inside $M_{n-1}(x)|\delta_x$ via the iteration strategy Σ which is guided by \mathcal{Q} -structures as in Definition 2.5.

In this case we can isolate the core model $K^{M_{n-1}(x)|\delta_x}$ below δ_x as in Theorem 1.1 in [JS13]. Then the core model $K^{M_{n-1}(x)|\delta_x}$ is absolute for all forcings of size less than δ_x over $M_{n-1}(x)|\delta_x$ and moreover $K^{M_{n-1}(x)|\delta_x}$ satisfies weak

covering using [MSch95]. That means we have that $M_{n-1}(x)|\delta_x \models “(\alpha^+)^K = \alpha^+”$ for all singular cardinals α .

Let $\alpha = \aleph_\omega^{M_{n-1}(x)}$. Then α is singular in $M_{n-1}(x)$, so we have in particular that

$$M_{n-1}(x)|\delta_x \models “(\alpha^+)^K = \alpha^+”.$$

Moreover we have that α is a cutpoint of $M_{n-1}(x)$. So let $z \in {}^\omega\omega$ be generic over $M_{n-1}(x)$ for $\text{Col}(\omega, \alpha)$. Then for $y = x \oplus z$ we have that

$$M_{n-1}(x)[z] = M_{n-1}(y),$$

where we construe $M_{n-1}^\#(x)[z]$ as a y -mouse and as a y -mouse $M_{n-1}^\#(x)[z]$ is sound and $\rho_\omega(M_{n-1}^\#(x)[z]) = y$ (see [SchSt09] for the fine structural details). Moreover we have that

$$M_{n-1}(y)|\delta_y \models \text{OD-determinacy},$$

since $y \geq_T x$. This implies that

$$M_{n-1}(x)[z]| \delta_y \models \text{OD-determinacy}.$$

Now work in the model $M_{n-1}(x)[z]| \delta_y$. Then we have that OD-determinacy implies that ω_1 is measurable in HOD as in Corollary 2.21.

Since $K \subseteq \text{HOD}$ and we have that $\omega_1 = (\alpha^+)^K$, it follows that $\omega_1 = (\alpha^+)^{\text{HOD}}$. But $\text{HOD} \models \text{AC}$, so in particular in HOD all measurable cardinals are inaccessible. This is a contradiction.

Case 2. Assume that there is a Woodin cardinal in $(K^c)^{M_{n-1}(x)|\delta_x}$.

In this case we aim to show that there exists a proper class inner model with n Woodin cardinals, which is obtained by performing a fully backgrounded extender construction inside $M_{n-1}(x)$ on top of the model

$$(K^c)^{M_{n-1}(x)|\delta_x} | \delta,$$

where δ denotes the largest Woodin cardinal in $(K^c)^{M_{n-1}(x)|\delta_x}$.

We can assume without loss of generality that there is a largest Woodin cardinal in the model $(K^c)^{M_{n-1}(x)|\delta_x}$ if it has a Woodin cardinal, because if there is no largest one, then $(K^c)^{M_{n-1}(x)|\delta_x}$ already yields a proper class inner model with n Woodin cardinals by iterating some large enough extender out of the universe. By the same argument we can in fact assume that $(K^c)^{M_{n-1}(x)|\delta_x}$ is $(n-1)$ -small above δ .

Let

$$(\mathcal{M}_\xi, \mathcal{N}_\xi \mid \xi \in \text{Ord})$$

be the sequence of models obtained from a fully backgrounded extender construction above $(K^c)^{M_{n-1}(x)|\delta_x} | \delta$ inside $M_{n-1}(x)$ in the sense of [MS94] but with the smallness hypothesis weakened where

$$\mathcal{M}_{\xi+1} = \mathcal{C}_\omega(\mathcal{N}_{\xi+1})$$

and let

$$L[E]((K^c)^{M_{n-1}(x)|\delta_x} | \delta)^{M_{n-1}(x)}$$

denote the resulting model.

Case 2.1. There is no $\xi \in \text{Ord}$ such that δ is not definably Woodin over the model $\mathcal{M}_{\xi+1}$.

In this case δ is a Woodin cardinal inside $L[E]((K^c)^{M_{n-1}(x)|\delta_x} | \delta)^{M_{n-1}(x)}$ and it follows by a generalization of Theorem 11.3 in [MS94] that we have that $L[E]((K^c)^{M_{n-1}(x)|\delta_x} | \delta)^{M_{n-1}(x)}$ is a proper class inner model with n Woodin cardinals, as desired.

Case 2.2. There exists a $\xi \in \text{Ord}$ such that δ is not definably Woodin over the model $\mathcal{M}_{\xi+1}$.

Let ξ be the minimal such ordinal. In this case the premouse $\mathcal{M}_{\xi+1}$ is $(n-1)$ -small above δ (see the proof of Claim 1 in the proof of Lemma 2.18) and we have that

$$\mathcal{M}_{\xi+1} \in M_{n-1}(x)|\delta_x.$$

Consider the coiteration of $\mathcal{M}_{\xi+1}$ and $(K^c)^{M_{n-1}(x)|\delta_x}$ inside $M_{n-1}(x)|\delta_x$.

Claim 1. *The coiteration of $\mathcal{M}_{\xi+1}$ and $(K^c)^{M_{n-1}(x)|\delta_x}$ inside $M_{n-1}(x)|\delta_x$ is successful.*

Proof. First of all we have that the coiteration takes place above δ and the premouse $\mathcal{M}_{\xi+1}$ is ω_1 -iterable above δ in V by construction (see [MS94]). Therefore the proof of Lemma 2.11 (2) yields that in the model $M_{n-1}(x)|\delta_x$ we have that $\mathcal{M}_{\xi+1}$ is iterable for iteration trees in $H_{\delta_x}^{M_{n-1}(x)}$ which are above δ , since $\mathcal{M}_{\xi+1} \in M_{n-1}(x)|\delta_x$ is $(n-1)$ -small above δ and $\rho_\omega(\mathcal{M}_{\xi+1}) \leq \delta$.

Moreover we have that $(K^c)^{M_{n-1}(x)|\delta_x}$ is countably iterable above δ inside $M_{n-1}(x)|\delta_x$ by the iterability proof in Chapter 9 in [St96].

Assume now toward a contradiction that the coiteration of $(K^c)^{M_{n-1}(x)|\delta_x}$ with $\mathcal{M}_{\xi+1}$ inside $M_{n-1}(x)|\delta_x$ is not successful. Since as argued above $\mathcal{M}_{\xi+1}$ is iterable above δ inside $M_{n-1}(x)|\delta_x$ and the coiteration takes place above δ this means that the coiteration has to fail on the $(K^c)^{M_{n-1}(x)|\delta_x}$ -side.

The premouse $(K^c)^{M_{n-1}(x)|\delta_x}$ is assumed to be $(n-1)$ -small above δ and therefore the fact that the coiteration of $(K^c)^{M_{n-1}(x)|\delta_x}$ and $\mathcal{M}_{\xi+1}$ fails on the $(K^c)^{M_{n-1}(x)|\delta_x}$ -side, implies that there exists an iteration tree \mathcal{T} on $(K^c)^{M_{n-1}(x)|\delta_x}$ of limit length such that there is no \mathcal{Q} -structure $\mathcal{Q}(\mathcal{T})$ for \mathcal{T} such that $\mathcal{Q}(\mathcal{T}) \trianglelefteq M_{n-2}^\#(\mathcal{M}(\mathcal{T}))$ and hence

$$M_{n-2}^\#(\mathcal{M}(\mathcal{T})) \vDash \text{“}\delta(\mathcal{T}) \text{ is Woodin”}.$$

In particular we have that the premouse $M_{n-2}^\#(\mathcal{M}(\mathcal{T}))$ constructed in the sense of Definition 2.9 is not $(n-2)$ -small above $\delta(\mathcal{T})$ since otherwise it

would already provide a \mathcal{Q} -structure $\mathcal{Q}(\mathcal{T})$ for \mathcal{T} which is $(n-2)$ -small above $\delta(\mathcal{T})$.

Let \bar{M} be the Mostowski collapse of a countable substructure of $M_{n-1}(x)|\delta_x$ containing the iteration tree \mathcal{T} . That means for a large enough natural number m we let \bar{M}, X and σ be such that

$$\bar{M} \cong X \prec_{\Sigma_m} M_{n-1}(x)|\delta_x,$$

where

$$\sigma : \bar{M} \rightarrow M_{n-1}(x)|\delta_x$$

denotes the uncollapse map such that we have a model \bar{K} in \bar{M} with $\sigma(\bar{K}|\gamma) = (K^c)^{M_{n-1}(x)|\delta_x}|\sigma(\gamma)$ for every ordinal $\gamma < \bar{M} \cap \text{Ord}$, and we have an iteration tree $\bar{\mathcal{T}}$ on \bar{K} in \bar{M} with $\sigma(\bar{\mathcal{T}}) = \mathcal{T}$. Moreover we let $\bar{\delta} \in \bar{M}$ be such that $\sigma(\bar{\delta}) = \delta$.

By the iterability proof of Chapter 9 in [St96] applied inside the model $M_{n-1}(x)|\delta_x$, there exists a cofinal well-founded branch b through the iteration tree $\bar{\mathcal{T}}$ on \bar{K} above $\bar{\delta}$. Moreover we have that

$$M_{n-2}^\#(\mathcal{M}(\bar{\mathcal{T}})) \models \text{“}\delta(\bar{\mathcal{T}}) \text{ is Woodin”}$$

and $M_{n-2}^\#(\mathcal{M}(\bar{\mathcal{T}}))$ is not $(n-2)$ -small above $\delta(\bar{\mathcal{T}})$.

Consider the coiteration of $\mathcal{M}_b^{\bar{\mathcal{T}}}$ with $M_{n-2}^\#(\mathcal{M}(\bar{\mathcal{T}}))$ and note that it takes place above $\delta(\bar{\mathcal{T}})$. Since $M_{n-2}^\#(\mathcal{M}(\bar{\mathcal{T}}))$ is ω_1 -iterable above $\delta(\bar{\mathcal{T}})$ and $\mathcal{M}_b^{\bar{\mathcal{T}}}$ is iterable above $\bar{\delta} < \delta(\bar{\mathcal{T}})$ by the iterability proof of Chapter 9 in [St96] applied inside $M_{n-1}(x)|\delta_x$, the coiteration is successful using Lemma 2.11 (2). We have that $\mathcal{M}_b^{\bar{\mathcal{T}}}$ cannot loose the coiteration by the following argument. If there is no drop along the branch b , then $\mathcal{M}_b^{\bar{\mathcal{T}}}$ cannot loose the coiteration, because then there is no definable Woodin cardinal in $\mathcal{M}_b^{\bar{\mathcal{T}}}$ above $\bar{\delta}$ by elementarity, but at the same time we have that

$$M_{n-2}^\#(\mathcal{M}(\bar{\mathcal{T}})) \models \text{“}\delta(\bar{\mathcal{T}}) > \bar{\delta} \text{ is Woodin”}.$$

If there is a drop along b , then $\mathcal{M}_b^{\bar{\mathcal{T}}}$ also has to win the coiteration, because we have that $\rho_\omega(\mathcal{M}_b^{\bar{\mathcal{T}}}) < \delta(\bar{\mathcal{T}})$ and $\rho_\omega(M_{n-2}^\#(\mathcal{M}(\bar{\mathcal{T}}))) = \delta(\bar{\mathcal{T}})$.

That means there is an iterate \mathcal{R}^* of $\mathcal{M}_b^{\bar{\mathcal{T}}}$ and a non-dropping iterate \mathcal{M}^* of $M_{n-2}^\#(\mathcal{M}(\bar{\mathcal{T}}))$ such that $\mathcal{M}^* \trianglelefteq \mathcal{R}^*$. We have that \mathcal{M}^* is not $(n-1)$ -small above $\bar{\delta}$, because $M_{n-2}^\#(\mathcal{M}(\bar{\mathcal{T}}))$ is not $(n-1)$ -small above $\bar{\delta}$ as argued above and the iteration from $M_{n-2}^\#(\mathcal{M}(\bar{\mathcal{T}}))$ to \mathcal{M}^* is non-dropping. Therefore it follows that \mathcal{R}^* is not $(n-1)$ -small above $\bar{\delta}$ and thus $\mathcal{M}_b^{\bar{\mathcal{T}}}$ is not $(n-1)$ -small above $\bar{\delta}$. By the iterability proof of Chapter 9 in [St96] we can re-embed the model $\mathcal{M}_b^{\bar{\mathcal{T}}}$ into a model of the $(K^c)^{M_{n-1}(x)|\delta_x}$ -construction above $(K^c)^{M_{n-1}(x)|\delta_x}|\delta$. This yields that $(K^c)^{M_{n-1}(x)|\delta_x}$ is not n -small, contradicting our assumption that it is $(n-1)$ -small above δ . \square

From Claim 1 it now follows by universality of $(K^c)^{M_{n-1}(x)|\delta_x}$ above δ (see Corollary 3.6 in [St96]) that the $(K^c)^{M_{n-1}(x)|\delta_x}$ -side has to win the comparison. That means there is an iterate K^* of $(K^c)^{M_{n-1}(x)|\delta_x}$ and an iterate N^* of $\mathcal{M}_{\xi+1}$ which is non-dropping on the main branch such that

$$N^* \trianglelefteq K^*.$$

But this is a contradiction, because we assumed that δ is not definably Woodin over $\mathcal{M}_{\xi+1}$ and at the same time we have that

$$(K^c)^{M_{n-1}(x)|\delta_x} \models \text{“}\delta \text{ is a Woodin cardinal”}.$$

This finishes the case that there is a Woodin cardinal in $(K^c)^{M_{n-1}(x)|\delta_x}$.

Case 3. Assume that there is no Woodin cardinal in $(K^c)^{M_{n-1}(x)|\delta_x}$ and that the premouse $(K^c)^{M_{n-1}(x)|\delta_x}$ is not fully iterable inside $M_{n-1}(x)|\delta_x$ via the iteration strategy Σ .

The failure of the attempt to iterate $(K^c)^{M_{n-1}(x)|\delta_x}$ via the \mathcal{Q} -structure iteration strategy Σ implies that there exists an iteration tree \mathcal{T} of limit length on $(K^c)^{M_{n-1}(x)|\delta_x}$ in $M_{n-1}(x)|\delta_x$ such that there exists no \mathcal{Q} -structure for \mathcal{T} inside the model $M_{n-1}(x)|\delta_x$.

Let

$$(\mathcal{M}_\xi, \mathcal{N}_\xi \mid \xi \in \text{Ord})$$

be the sequence of models obtained from a fully backgrounded extender construction above $\mathcal{M}(\mathcal{T})$ inside $M_{n-1}(x)$ in the sense of [MS94] but with the smallness hypothesis weakened where

$$\mathcal{M}_{\xi+1} = \mathcal{C}_\omega(\mathcal{N}_{\xi+1})$$

and let

$$L[E](\mathcal{M}(\mathcal{T}))^{M_{n-1}(x)}$$

denote the resulting model.

Case 3.1. There is no $\xi \in \text{Ord}$ such that $\delta(\mathcal{T})$ is not definably Woodin over the model $\mathcal{M}_{\xi+1}$.

In this case $\delta(\mathcal{T})$ is a Woodin cardinal inside $L[E](\mathcal{M}(\mathcal{T}))^{M_{n-1}(x)}$ and it follows as in Case 2.1 by a generalization of Theorem 11.3 in [MS94] that $L[E](\mathcal{M}(\mathcal{T}))^{M_{n-1}(x)}$ is a proper class inner model with n Woodin cardinals, as desired.

Case 3.2. There exists a $\xi \in \text{Ord}$ such that $\delta(\mathcal{T})$ is not definably Woodin over the model $\mathcal{M}_{\xi+1}$.

Let ξ be the minimal such ordinal. In this case the premouse $\mathcal{M}_{\xi+1}$ is $(n-1)$ -small above $\delta(\mathcal{T})$ and we have that

$$\mathcal{M}_{\xi+1} \in M_{n-1}(x)|\delta_x.$$

But then $\mathcal{M}_{\xi+1} \triangleright \mathcal{M}(\mathcal{T})$ already provides a \mathcal{Q} -structure for \mathcal{T} inside the model $M_{n-1}(x)|\delta_x$ because $\delta(\mathcal{T})$ is not definably Woodin over $\mathcal{M}_{\xi+1}$. This is a contradiction. \square

Note that all results we proved in this section under a lightface determinacy hypothesis relativize to all $x \in {}^\omega\omega$ if we assume the analogous boldface determinacy hypothesis. We just decided to present the results without additional parameters to simplify the notation.

3. PROVING ITERABILITY

With Theorem 2.24 we found a candidate for M_n in the previous section, but we still have to show its iterability. We will in fact not prove that this candidate is iterable, but we will use it to construct an ω_1 -iterable premouse $M_n^\#$ in the case that n is odd. (Note here already that we will give a different argument if n is even.)

We do parts of this in a slightly more general context and therefore introduce the concept of an n -suitable premouse in Section 3.1, which will be a natural candidate for the premouse $M_n|(\delta_0^+)^{M_n}$, where δ_0 denotes the least Woodin cardinal in M_n . Using n -suitable premice we will show inductively in Sections 3.5 and 3.6 that under a determinacy hypothesis $M_n^\#$ exists and is ω_1 -iterable.

In this section again all results we are going to prove under a lightface determinacy hypothesis relativize to all $x \in {}^\omega\omega$ under the analogous boldface determinacy hypothesis.

3.1. Existence of n -suitable Premice. After introducing pre- n -suitable premice and proving their existence from the results in the previous section, we aim to show in this section that pre- $(2n-1)$ -suitable premice, which are premice with one Woodin cardinal which satisfy certain fullness conditions, also satisfy a weak form of iterability, namely short tree iterability. In fact we are going to show a slightly stronger form of iterability which includes that fullness properties are preserved during non-dropping iterations. This will in particular enable us to perform certain comparison arguments for $(2n-1)$ -small premice and will therefore help us to conclude ω_1 -iterability for some candidate for $M_{2n-1}^\#$.

Recall that in what follows by “ $M_n^\#$ exists” we always mean that “ $M_n^\#$ exists and is ω_1 -iterable”.

A lot of the results in this section only hold true for premice at the odd levels of our argument, namely $(2n-1)$ -suitable premice. This results from the periodicity in the projective hierarchy in terms of the uniformization property (see Theorem 6C.5 in [Mo09]) and the periodicity in the correctness of $M_n^\#$ (see Lemmas 1.17 and 1.19). This behaviour forces us to give a different proof for the even levels of our argument in Section 3.6.

We start by introducing pre- n -suitable and n -suitable premice. Our definition will generalize the notion of suitability from Definition 3.4 in [StW16] to $n > 1$. For technical reasons our notion slightly differs from n -suitability as defined in Definition 5.2 in [Sa13].

Definition 3.1. *Let $n \geq 1$ and assume that $M_{n-1}^\#(x)$ exists for all $x \in {}^\omega\omega$. Then we say a countable premouse N is pre- n -suitable iff there is an ordinal $\delta < \omega_1^V$ such that*

(1) $N \models \text{“ZFC}^- + \delta \text{ is the largest cardinal”}$,

$$N = M_{n-1}(N|\delta) | (\delta^+)^{M_{n-1}(N|\delta)},$$

and for every $\gamma < \delta$,

$$M_{n-1}(N|\gamma) | (\gamma^+)^{M_{n-1}(N|\gamma)} \triangleleft N,$$

(2) $M_{n-1}(N|\delta)$ is a proper class model and

$$M_{n-1}(N|\delta) \models \text{“}\delta \text{ is Woodin”},$$

(3) for every $\gamma < \delta$, $M_{n-1}(N|\gamma)$ is a set, or

$$M_{n-1}(N|\gamma) \not\models \text{“}\gamma \text{ is Woodin”},$$

and

(4) for every $\eta < \delta$, $M_{n-1}(N|\delta) \models \text{“}N|\delta \text{ is } \eta\text{-iterable”}$.

Recall the definition of the premouse $M_{n-1}(N|\delta)$ from Definition 2.9. If N is a pre- n -suitable premouse, we denote the unique ordinal δ from Definition 3.1 by δ_N , analogous to the notation fixed in Section 2.1.

Whenever we assume that some premouse N is pre- n -suitable for some $n \geq 1$, we in fact tacitly assume in addition that the premouse $M_{n-1}^\#(x)$ exists for all $x \in {}^\omega\omega$ (or at least that the premouse $M_{n-1}^\#(N|\delta)$ exists).

Remark. Clearly, if it exists, $M_n|(\delta^+)^{M_n}$ is a pre- n -suitable premouse for $n \geq 1$, whenever δ denotes the least Woodin cardinal in M_n .

Remark. We have that for $n \geq 1$, if N is a pre- n -suitable premouse, then N is n -small.

We first show that the proper class inner model with n Woodin cardinals we constructed in the proof of Theorem 2.24 yields a pre- n -suitable premouse, if we cut it off at the successor of its least Woodin cardinal and minimize it.

Lemma 3.2. *Let $n \geq 1$. Assume that $M_{n-1}^\#(x)$ exists for all $x \in {}^\omega\omega$ and that all Σ_{n+1}^1 -definable sets of reals are determined. Then there exists a pre- n -suitable premouse.*

Proof. Let W be the model constructed in Cases 2 and 3 in the proof of Theorem 2.24. Cutting off at the successor of the bottom Woodin cardinal δ yields a premouse $N = W|(\delta^+)^W$ which satisfies conditions (1) and (2) in the definition of pre- n -suitability, in the case that the premouse $(K^c)^{M_{n-1}(x)|\delta_x}$

from the proof of Theorem 2.24 is $(n - 1)$ -small above δ . Otherwise we can easily consider an initial segment N of $(K^c)^{M_{n-1}(x)|\delta_x}$ which satisfies conditions (1) and (2). Let N' be the minimal initial segment of N which satisfies conditions (1) and (2) and note that we then have that N' satisfies condition (3).

Now the iterability proof from Chapter 9 in [St96] shows that this premouse $N'|\delta'$ is countably iterable inside $M_{n-1}(x)$, where x is a real as in the proof of Theorem 2.24 such that the model W as above is constructed inside the model $M_{n-1}(x)$ and δ' denotes the largest cardinal in N' . The \mathcal{Q} -structures for iteration trees \mathcal{T} on $N'|\delta'$ are $(n - 1)$ -small above the common part model and are therefore contained in the model $M_{n-1}(N'|\delta')$ by arguments we already gave several times before. Thus it follows that $N'|\delta'$ is η -iterable inside $M_{n-1}(N'|\delta')$ for all $\eta < \delta'$. Therefore we have that condition (4) holds as well for N' . \square

We can show that the following weak form of condensation holds for pre- n -suitable premice.

Lemma 3.3 (Weak Condensation Lemma). *Let N be a pre- n -suitable premouse for some $n \geq 1$ and let δ_N denote the largest cardinal in N . Let γ be a large enough countable ordinal in V and let H be the Mostowski collapse of $\text{Hull}_m^{M_{n-1}(N|\delta_N)|\gamma}(\{\delta_N\})$ for some large enough natural number m . Then*

$$H \triangleleft M_{n-1}(N|\delta_N).$$

Proof. Consider the coiteration of the premice $N|\delta_N$ and H inside the model $M_{n-1}(N|\delta_N)$. By condition (4) in the definition of pre- n -suitability we have that $N|\delta_N$ is iterable enough for the comparison with H inside the model $M_{n-1}(N|\delta_N)$. Moreover H is also iterable enough for the comparison via the realization strategy (see Section 4 in [MaSt94]). So the comparison is successful and there are iterates N^* of $N|\delta_N$ and H^* of H such that $H^* \trianglelefteq N^*$ or $N^* \trianglelefteq H^*$. The Weak Dodd-Jensen Lemma (see Theorem 4.10 in [St10]) yields that in fact $H^* \trianglelefteq N^*$ and that the iteration from H to H^* does not drop.

Assume toward a contradiction that $H \neq H^*$, that means there is at least one extender used in the iteration from H to H^* . Since $\rho_\omega(H) = \omega$, this implies that H^* is not sound and yields a contradiction to the fact that we have $H^* \trianglelefteq N^*$. So it follows that H is not moved in the coiteration and therefore $H = H^*$.

Assume now that the premouse $N|\delta_N$ moves in the coiteration, that means we have that $N^* \neq N|\delta_N$. Let E_α for some ordinal α be the first extender used in the iteration from $N|\delta_N$ to N^* . Then the index α of the extender E_α is a cardinal in N^* . Since we have $\rho_\omega(H) = \omega$ and $H \trianglelefteq N^*$, it follows that $\alpha > H \cap \text{Ord}$. Therefore there was no need to iterate $N|\delta_N$ at all and we have that

$$H \trianglelefteq N|\delta_N \triangleleft M_{n-1}(N|\delta_N),$$

as desired. \square

Analogous to Definitions 3.6 and 3.9 in [StW16] we define a notion of short tree iterability for pre- n -suitable premouse. Informally a pre- n -suitable premouse is short tree iterable if it is iterable with respect to iteration trees for which there are \mathcal{Q} -structures (see Definition 2.4) which are not too complicated. For this definition we again tacitly assume that $M_{n-1}^\#(x)$ exists for all reals x .

Definition 3.4. *Let \mathcal{T} be a normal iteration tree of length $< \omega_1^V$ on a pre- n -suitable premouse N for some $n \geq 1$. We say \mathcal{T} is short iff for all limit ordinals $\lambda < \text{lh}(\mathcal{T})$ the \mathcal{Q} -structure $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ exists, is $(n-1)$ -small above $\delta(\mathcal{T} \upharpoonright \lambda)$ and we have that*

$$\mathcal{Q}(\mathcal{T} \upharpoonright \lambda) \trianglelefteq \mathcal{M}_\lambda^{\mathcal{T}},$$

and if \mathcal{T} has limit length we in addition have that $\mathcal{Q}(\mathcal{T})$ exists and

$$\mathcal{Q}(\mathcal{T}) \trianglelefteq M_{n-1}(\mathcal{M}(\mathcal{T})).$$

Moreover we say \mathcal{T} is maximal iff \mathcal{T} is not short.

The premouse $M_{n-1}(\mathcal{M}(\mathcal{T}))$ in Definition 3.4 is defined as in Definition 2.9.

Definition 3.5. *Let N be a pre- n -suitable premouse for some $n \geq 1$. We say N is short tree iterable iff whenever \mathcal{T} is a short tree on N ,*

- (i) *if \mathcal{T} has a last model, then every putative² iteration tree \mathcal{U} extending \mathcal{T} such that $\text{lh}(\mathcal{U}) = \text{lh}(\mathcal{T}) + 1$ has a well-founded last model, and*
- (ii) *if \mathcal{T} has limit length, then there exists a unique cofinal well-founded branch b through \mathcal{T} such that*

$$\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T}).$$

Remark. At this point in contrast to the notion of short tree iterability for 1-suitable premouse in [StW16] we do not require the iterate of a pre- n -suitable premouse via a short tree to be pre- n -suitable again. The reason for this is that in the general case for $n > 1$ it is not obvious that this property holds assuming only our notion of short tree iterability as defined above. We will be able to prove later in Lemma 3.9 that this property in fact does hold true.

Because of the periodicity in the projective hierarchy (see also [St95] for the periodicity in the definition of Π_n^1 -iterability) the proof of the following lemma only works for odd levels of suitability.

²Recall that we say an iteration tree \mathcal{U} is a *putative iteration tree* if \mathcal{U} satisfies all properties of an iteration tree, but we allow the last model of \mathcal{U} to be ill-founded, in case \mathcal{U} has a last model.

Lemma 3.6. *Let $n \geq 0$ and assume that $M_{2n}^\#(x)$ exists for all $x \in {}^\omega\omega$. Let N be a pre- $(2n+1)$ -suitable premouse. Then the statement “ N is short tree iterable” as in Definition 3.5 is Π_{2n+2}^1 -definable uniformly in any code for the countable premouse N .*

Proof. The statement “ N is short tree iterable” can be phrased as follows. We first consider trees of limit length.

$\forall \mathcal{T}$ tree on N of limit length $\forall (\mathcal{Q}_\lambda \mid \lambda \leq \text{lh}(\mathcal{T}) \text{ limit ordinal})$,
 if for all limit ordinals $\lambda \leq \text{lh}(\mathcal{T})$,
 \mathcal{Q}_λ is Π_{2n+1}^1 -iterable above $\delta(\mathcal{T} \upharpoonright \lambda)$, $2n$ -small above $\delta(\mathcal{T} \upharpoonright \lambda)$,
 solid above $\delta(\mathcal{T} \upharpoonright \lambda)$ and a \mathcal{Q} -structure for $\mathcal{T} \upharpoonright \lambda$, and
 if for all limit ordinals $\lambda < \text{lh}(\mathcal{T})$ we have $\mathcal{Q}_\lambda \trianglelefteq \mathcal{M}_\lambda^\mathcal{T}$, then
 $\exists b$ cofinal branch through \mathcal{T} such that $\mathcal{Q}_{\text{lh}(\mathcal{T})} \trianglelefteq \mathcal{M}_b^\mathcal{T}$.

This statement is Π_{2n+2}^1 -definable uniformly in any code for N since Π_{2n+1}^1 -iterability above $\delta(\mathcal{T} \upharpoonright \lambda)$ for \mathcal{Q}_λ is Π_{2n+1}^1 -definable uniformly in any code for \mathcal{Q}_λ . For trees of successor length we get a similar statement as follows.

$\forall \mathcal{T}$ putative tree on N of successor length $\forall (\mathcal{Q}_\lambda \mid \lambda < \text{lh}(\mathcal{T}) \text{ limit ordinal})$,
 if for all limit ordinals $\lambda < \text{lh}(\mathcal{T})$,
 \mathcal{Q}_λ is Π_{2n+1}^1 -iterable above $\delta(\mathcal{T} \upharpoonright \lambda)$, $2n$ -small above $\delta(\mathcal{T} \upharpoonright \lambda)$,
 solid above $\delta(\mathcal{T} \upharpoonright \lambda)$ and a \mathcal{Q} -structure for $\mathcal{T} \upharpoonright \lambda$, and
 if for all limit ordinals $\lambda < \text{lh}(\mathcal{T})$ we have $\mathcal{Q}_\lambda \trianglelefteq \mathcal{M}_\lambda^\mathcal{T}$, then
 the last model of \mathcal{T} is well-founded.

As above this statement is also Π_{2n+2}^1 -definable uniformly in any code for N . Moreover the conjunction of these two statements is equivalent to the statement “ N is short tree iterable”, because the relevant \mathcal{Q} -structures \mathcal{Q}_λ for limit ordinals $\lambda \leq \text{lh}(\mathcal{T})$ are $2n$ -small above $\delta(\mathcal{T} \upharpoonright \lambda)$ and thus Lemma 2.13 implies that for them it is enough to demand Π_{2n+1}^1 -iterability above $\delta(\mathcal{T} \upharpoonright \lambda)$ to identify them as a \mathcal{Q} -structure for $\mathcal{T} \upharpoonright \lambda$ since we assumed that $M_{2n}^\#(x)$ exists for all $x \in {}^\omega\omega$. \square

From this we can obtain the following corollary using Lemma 1.17.

Corollary 3.7. *Let $n \geq 0$ and assume that $M_{2n}^\#(x)$ exists for all $x \in {}^\omega\omega$. If N is a pre- $(2n+1)$ -suitable premouse, then N is short tree iterable iff N is short tree iterable inside the model $M_{2n}(N \mid \delta_N)^{\text{Col}(\omega, \delta_N)}$, where δ_N again denotes the largest cardinal in N .*

Proof. Let N be an arbitrary pre- $(2n+1)$ -suitable premouse. By Lemma 3.6 we have that short tree iterability for N is a Π_{2n+2}^1 -definable statement uniformly in any code for N . Therefore we have by Lemma 1.17 that N is short tree iterable inside the model $M_{2n}(N \mid \delta_N)^{\text{Col}(\omega, \delta_N)}$ iff N is short tree

iterable in V , because the model $M_{2n}(N|\delta_N)^{\text{Col}(\omega, \delta_N)}$ is Σ_{2n+2}^1 -correct in V . \square

In what follows we aim to show that every pre- $(2n+1)$ -suitable premouse N is short tree iterable. In fact we are going to show a stronger form of iterability for pre- $(2n+1)$ -suitable premice, including for example fullness-preservation for short trees. This means that for non-dropping short trees \mathcal{T} on N of length $\lambda+1$ for some ordinal $\lambda < \omega_1^V$ the resulting model of the iteration $\mathcal{M}_\lambda^\mathcal{T}$ is again pre- $(2n+1)$ -suitable. Here we mean by “non-dropping” that the tree \mathcal{T} does not drop on the main branch $[0, \lambda]_{\mathcal{T}}$. If this property holds for a pre- $(2n+1)$ -suitable premouse N we say that N has a *fullness preserving iteration strategy for short trees*. Moreover we also want to show some form of iterability including fullness-preservation for maximal trees on N . Premice which satisfy all these kinds of iterability we will call $(2n+1)$ -suitable.

The exact form of iterability we are aiming for is introduced in the following definition.

Definition 3.8. *Assume that $M_{n-1}^\#(x)$ exists for all $x \in {}^\omega\omega$ and let N be a pre- n -suitable premouse for some $n \geq 1$. Then we say that the premouse N is n -suitable iff*

- (i) N is short tree iterable and whenever \mathcal{T} is a short tree on N of length $\lambda+1$ for some ordinal $\lambda < \omega_1^V$ which is non-dropping on the main branch $[0, \lambda]_{\mathcal{T}}$, then the final model $\mathcal{M}_\lambda^\mathcal{T}$ is pre- n -suitable, and
- (ii) whenever \mathcal{T} is a maximal iteration tree on N of length λ for some limit ordinal $\lambda < \omega_1^V$ according to the \mathcal{Q} -structure iteration strategy, then there exists a cofinal well-founded branch b through \mathcal{T} such that b is non-dropping and the model $\mathcal{M}_b^\mathcal{T}$ is pre- n -suitable. In fact we have in this case that

$$\mathcal{M}_b^\mathcal{T} = M_{n-1}(\mathcal{M}(\mathcal{T}))|(\delta(\mathcal{T})^+)^{M_{n-1}(\mathcal{M}(\mathcal{T}))}.$$

Now we are ready to prove that every pre- $(2n+1)$ -suitable premouse is in fact already $(2n+1)$ -suitable, using the iterability we build into condition (4) of Definition 3.1 in form of the Weak Condensation Lemma (see Lemma 3.3).

Lemma 3.9. *Let $n \geq 0$ and assume that $M_{2n}^\#(x)$ exists for all $x \in {}^\omega\omega$. Let N be an arbitrary pre- $(2n+1)$ -suitable premouse. Then N is $(2n+1)$ -suitable.*

Proof. Let N be an arbitrary pre- $(2n+1)$ -suitable premouse and let $W = M_{2n}(N|\delta_N)$ be a premouse in the sense of Definition 2.9, where δ_N as usual denotes the largest cardinal in N . That means in particular that $N = W|(\delta_N^+)^W$.

We want to show that N is $(2n + 1)$ -suitable. So we assume toward a contradiction that this is not the case. Say this is witnessed by an iteration tree \mathcal{T} on N .

We want to reflect this statement down to a countable hull. Therefore let m be a large enough natural number, let θ be a large enough ordinal such that in particular

$$W|\theta \prec_{\Sigma_m} W$$

and

$$W|\theta \models \text{ZFC}^-,$$

and let

$$\bar{W} \stackrel{\sigma}{\cong} \text{Hull}_m^{W|\theta}(\{\delta_N\}) \prec W|\theta,$$

where \bar{W} is the Mostowski collapse of $\text{Hull}_m^{W|\theta}(\{\delta_N\})$ and

$$\sigma : \bar{W} \rightarrow \text{Hull}_m^{W|\theta}(\{\delta_N\})$$

denotes the uncollapse map such that $\delta_N \in \text{ran}(\sigma)$ and $\sigma(\bar{\delta}) = \delta_N$ for some ordinal $\bar{\delta}$ in \bar{W} . Then we have that \bar{W} is sound, $\rho_{m+1}(\bar{W}) = \omega$, and the Weak Condensation Lemma 3.3 yields that

$$\bar{W} \triangleleft W.$$

Case 1. \mathcal{T} is short and witnesses that N is not short tree iterable.

For simplicity assume in this case that \mathcal{T} has limit length since the other case is easier. Then \mathcal{T} witnesses that the following statement $\phi_1(N)$ holds in V .

$\phi_1(N) \equiv \exists \mathcal{T}$ tree on N of length λ for some limit ordinal $\lambda < \omega_1^V$
 $\exists (\mathcal{Q}_\gamma \mid \gamma \leq \lambda \text{ limit ordinal})$, such that for all limit ordinals $\gamma \leq \lambda$,
 \mathcal{Q}_γ is Π_{2n+1}^1 -iterable above $\delta(\mathcal{T} \upharpoonright \gamma)$, $2n$ -small above $\delta(\mathcal{T} \upharpoonright \gamma)$,
solid above $\delta(\mathcal{T} \upharpoonright \gamma)$ and a \mathcal{Q} -structure for $\mathcal{T} \upharpoonright \gamma$, and
for all limit ordinals $\gamma < \lambda$ we have $\mathcal{Q}_\gamma \trianglelefteq \mathcal{M}_\gamma^{\mathcal{T}}$, but
there exists no cofinal branch b through \mathcal{T} such that $\mathcal{Q}_\lambda \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$.

We have that $\phi_1(N)$ is Σ_{2n+2}^1 -definable uniformly in any code for N as in the proof of Lemma 3.6. See also the proof of Lemma 3.6 for the case that \mathcal{T} has successor length.

Case 2. \mathcal{T} is a short tree on N of length $\lambda + 1$ for some ordinal $\lambda < \omega_1^V$ which is non-dropping on the main branch such that the final model $\mathcal{M}_\lambda^{\mathcal{T}}$ is not pre- $(2n + 1)$ -suitable.

Assume that

$$M_{2n}(\mathcal{M}_\lambda^{\mathcal{T}} \mid \delta_{\mathcal{M}_\lambda^{\mathcal{T}}}) \not\models \text{“}\delta_{\mathcal{M}_\lambda^{\mathcal{T}}} \text{ is Woodin”},$$

where $\delta_{\mathcal{M}_\lambda^\mathcal{T}}$ denotes the largest cardinal in $\mathcal{M}_\lambda^\mathcal{T}$. This means that we have $\delta_{\mathcal{M}_\lambda^\mathcal{T}} = i_{0\lambda}^\mathcal{T}(\delta_N)$, where $i_{0\lambda}^\mathcal{T} : N \rightarrow \mathcal{M}_\lambda^\mathcal{T}$ denotes the iteration embedding, which exists since the iteration tree \mathcal{T} is assumed to be non-dropping on the main branch.

Then \mathcal{T} witnesses that the following statement $\phi_2(N)$ holds true in V .

$$\begin{aligned} \phi_2(N) \equiv & \exists \mathcal{T} \text{ tree on } N \text{ of length } \lambda + 1 \text{ for some } \lambda < \omega_1^V \text{ such that} \\ & \mathcal{T} \text{ is non-dropping along } [0, \lambda]_{\mathcal{T}} \text{ and} \\ & \forall \gamma < \text{lh}(\mathcal{T}) \text{ limit } \exists \mathcal{Q} \trianglelefteq \mathcal{M}_\gamma^\mathcal{T} \text{ such that} \\ & \quad \mathcal{Q} \text{ is } \Pi_{2n+1}^1\text{-iterable above } \delta(\mathcal{T} \upharpoonright \gamma), 2n\text{-small above } \delta(\mathcal{T} \upharpoonright \gamma), \\ & \quad \text{solid above } \delta(\mathcal{T} \upharpoonright \gamma), \text{ and a } \mathcal{Q}\text{-structure for } \mathcal{T} \upharpoonright \gamma, \text{ and} \\ & \quad \exists \mathcal{P} \triangleright \mathcal{M}_\lambda^\mathcal{T} | \delta_{\mathcal{M}_\lambda^\mathcal{T}} \text{ such that } \mathcal{P} \text{ is } \Pi_{2n+1}^1\text{-iterable above } i_{0\lambda}^\mathcal{T}(\delta_N), \\ & \quad 2n\text{-small above } i_{0\lambda}^\mathcal{T}(\delta_N), i_{0\lambda}^\mathcal{T}(\delta_N)\text{-sound, } \rho_\omega(\mathcal{P}) \leq i_{0\lambda}^\mathcal{T}(\delta_N), \\ & \quad \text{and } i_{0\lambda}^\mathcal{T}(\delta_N) \text{ is not definably Woodin over } \mathcal{P}, \end{aligned}$$

where δ_N as above denotes the largest cardinal in N . Recall Definition 2.7 for the notion of a definable Woodin cardinal.

We have that $\phi_2(N)$ is Σ_{2n+2}^1 -definable uniformly in any code for N .

Case 3. \mathcal{T} is a maximal tree on N of length λ for some limit ordinal $\lambda < \omega_1^V$ such that there is no cofinal well-founded branch b through \mathcal{T} or for every such branch b the premouse $\mathcal{M}_b^\mathcal{T}$ is not pre- $(2n+1)$ -suitable.

As \mathcal{T} is maximal, we have that every such branch b is non-dropping and in the case that for every such branch b the premouse $\mathcal{M}_b^\mathcal{T}$ is not pre- $(2n+1)$ -suitable, assume that we have

$$M_{2n}(\mathcal{M}_b^\mathcal{T} | \delta_{\mathcal{M}_b^\mathcal{T}}) \not\models \text{“}\delta_{\mathcal{M}_b^\mathcal{T}} \text{ is Woodin”},$$

where $\delta_{\mathcal{M}_b^\mathcal{T}}$ denotes the largest cardinal in $\mathcal{M}_b^\mathcal{T}$. Then the iteration tree \mathcal{T} witnesses that the following statement $\phi_3(N)$ holds true in V .

$$\begin{aligned} \phi_3(N) \equiv & \exists \mathcal{T} \text{ tree on } N \text{ of length } \lambda \text{ for some limit ordinal } \lambda < \omega_1^V \text{ such that} \\ & \forall \gamma < \lambda \text{ limit } \exists \mathcal{Q} \trianglelefteq \mathcal{M}_\gamma^\mathcal{T} \text{ such that} \\ & \quad \mathcal{Q} \text{ is } \Pi_{2n+1}^1\text{-iterable above } \delta(\mathcal{T} \upharpoonright \gamma), 2n\text{-small above } \delta(\mathcal{T} \upharpoonright \gamma), \\ & \quad \text{solid above } \delta(\mathcal{T} \upharpoonright \gamma), \text{ and a } \mathcal{Q}\text{-structure for } \mathcal{T} \upharpoonright \gamma, \text{ and} \\ & \quad \exists \mathcal{P} \triangleright \mathcal{M}(\mathcal{T}) \text{ such that } \mathcal{P} \text{ is } \Pi_{2n+1}^1\text{-iterable above } \delta(\mathcal{T}), \\ & \quad \rho_\omega(\mathcal{P}) \leq \delta(\mathcal{T}), \mathcal{P} \text{ is } 2n\text{-small above } \delta(\mathcal{T}), \delta(\mathcal{T})\text{-sound, and} \\ & \quad \mathcal{P} \models \text{“ZFC}^- + \delta(\mathcal{T}) \text{ is the largest cardinal} + \delta(\mathcal{T}) \text{ is Woodin”}, \\ & \quad \text{and there is no branch } b \text{ through } \mathcal{T} \text{ such that } \mathcal{P} \trianglelefteq \mathcal{M}_b^\mathcal{T}. \end{aligned}$$

We have that $\phi_3(N)$ is Σ_{2n+2}^1 -definable uniformly in any code for N .

Now we consider all three cases together again and let

$$\phi(N) = \phi_1(N) \vee \phi_2(N) \vee \phi_3(N).$$

As argued in the individual cases the iteration tree \mathcal{T} witnesses that $\phi(N)$ holds in V (still assuming for simplicity that \mathcal{T} has limit length if \mathcal{T} is as in Case 1).

Since ϕ is a Σ_{2n+2}^1 -definable statement and $W = M_{2n}(N|\delta_N)$, we have by Lemma 1.17 that

$$\Vdash_{\text{Col}(\omega, \delta_N)}^W \phi(N).$$

So since we picked m and θ large enough we have that

$$\bar{W}[g] \models \phi(\bar{N}),$$

if we let $\bar{N} \in \bar{W}$ be such that $\sigma(\bar{N}) = N$ and if g is $\text{Col}(\omega, \bar{\delta})$ -generic over \bar{W} . Let $\bar{\mathcal{T}}$ be a tree on \bar{N} in $\bar{W}[g]$ witnessing that $\phi(\bar{N})$ holds. Since \bar{N} is countable in W , we can pick $g \in W$ and then have that $\bar{\mathcal{T}} \in W$.

We have that $\bar{W}[g]$ is Σ_{2n+1}^1 -correct in V using Lemma 1.18, because it is a countable model with $2n$ Woodin cardinals. Since $\bar{\mathcal{T}}$ witnesses the statement $\phi(\bar{N})$ in $\bar{W}[g]$, it follows that $\bar{\mathcal{T}}$ also witnesses $\phi(\bar{N})$ in V , because $\phi(\bar{N})$ is Σ_{2n+2}^1 -definable in any code for \bar{N} . The \mathcal{Q} -structures for $\bar{\mathcal{T}}$ in $\bar{W}[g]$ in the statement $\phi(\bar{N})$ are Π_{2n+1}^1 -iterable above $\delta(\bar{\mathcal{T}} \upharpoonright \gamma)$ and $2n$ -small above $\delta(\bar{\mathcal{T}} \upharpoonright \gamma)$ for limit ordinals $\gamma < \text{lh}(\bar{\mathcal{T}})$ (and also for $\gamma = \text{lh}(\bar{\mathcal{T}})$ if $\bar{\mathcal{T}}$ witnesses that $\phi_1(\bar{N})$ holds in $\bar{W}[g]$).

Since this amount of iterability suffices to witness \mathcal{Q} -structures using Lemma 2.13 and since as mentioned above $\bar{W}[g]$ is Σ_{2n+1}^1 -correct in V , the \mathcal{Q} -structures for $\bar{\mathcal{T}}$ in $\bar{W}[g]$ are also \mathcal{Q} -structures for $\bar{\mathcal{T}}$ inside V . Since $W = M_{2n}(N|\delta_N)$ is also Σ_{2n+1}^1 -correct in V using Lemma 1.17 and \bar{N} and $\bar{\mathcal{T}}$ are countable in W , it follows that the \mathcal{Q} -structures for $\bar{\mathcal{T}}$ in $\bar{W}[g]$ (which are Π_{2n+1}^1 -iterable above $\delta(\bar{\mathcal{T}} \upharpoonright \gamma)$ for γ as above) are also \mathcal{Q} -structures for $\bar{\mathcal{T}}$ inside W . Therefore the branches chosen in the tree $\bar{\mathcal{T}}$ on \bar{N} inside $\bar{W}[g]$ are the same branches as the \mathcal{Q} -structure iteration strategy Σ as in Definition 2.5 would choose inside the model W when iterating the premouse \bar{W} . That means if \mathcal{T}^* is the tree on \bar{W} obtained by considering $\bar{\mathcal{T}}$ as a tree on $\bar{W} \triangleright \bar{N}$, then the iteration strategy Σ picks the same branches for the tree \mathcal{T}^* as it does for the tree $\bar{\mathcal{T}}$.

Now we again distinguish three cases as before.

Case 1. $\bar{\mathcal{T}}$ witnesses that $\phi_1(\bar{N})$ holds in $\bar{W}[g]$.

By the argument we gave above, $\bar{\mathcal{T}}$ is a short tree on \bar{N} in W .

Let \bar{b} denote the cofinal branch through \mathcal{T}^* which exists inside W and is defined as follows. We have that a branch through the iteration tree \mathcal{T}^* can be considered as a branch through $\bar{\mathcal{T}}$, where the latter is a tree on \bar{N} , and vice versa. Since by the Weak Condensation Lemma 3.3 we have that

$$\bar{N} = \bar{W} | (\bar{\delta}^+)^{\bar{W}} \triangleleft W | \omega_1^W = M_{2n}(N|\delta_N) | \omega_1^{M_{2n}(N|\delta_N)} = N | \omega_1^N,$$

there exists a cofinal well-founded branch $\bar{b} \in W$ through $\bar{\mathcal{T}}$ by property (4) in the definition of pre- n -suitability (see Definition 3.1). We also consider this branch \bar{b} as a branch through \mathcal{T}^* .

Assume first that there is a drop along the branch \bar{b} . Then there exists a \mathcal{Q} -structure $\mathcal{Q}(\bar{\mathcal{T}}) \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$. Consider the statement

$$\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}})) \equiv \text{“there is a cofinal branch } b \text{ through } \bar{\mathcal{T}} \text{ such that } \mathcal{Q}(\bar{\mathcal{T}}) \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}} \text{”}.$$

This statement $\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}}))$ is Σ_1^1 -definable uniformly in any code for the parameters $\bar{\mathcal{T}}$ and $\mathcal{Q}(\bar{\mathcal{T}})$ and holds in the model W as witnessed by the branch \bar{b} . Now a Σ_1^1 -absoluteness argument as the one given in the proof of Lemma 2.11 yields that this statement $\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}}))$ also holds in $\bar{W}[g]$, which contradicts the fact that $\bar{\mathcal{T}}$ witnesses in $\bar{W}[g]$ that \bar{N} is not short tree iterable.

Therefore we can assume that \bar{b} does not drop.

Since $\bar{\mathcal{T}}$ witnesses that $\phi_1(\bar{N})$ holds in $\bar{W}[g]$, we have that there exists a \mathcal{Q} -structure \mathcal{Q}_λ for $\bar{\mathcal{T}}$ as in $\phi_1(\bar{N})$. In particular \mathcal{Q}_λ is $2n$ -small above $\delta(\mathcal{T})$ and Π_{2n+1}^1 -iterable above $\delta(\mathcal{T})$ in $\bar{W}[g]$.

Case 1.1. $\delta(\bar{\mathcal{T}}) = i_{\bar{b}}^{\bar{\mathcal{T}}^*}(\bar{\delta})$.

Consider the comparison of \mathcal{Q}_λ with $\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*}$ inside W .

This comparison takes place above $i_{\bar{b}}^{\bar{\mathcal{T}}^*}(\bar{\delta}) = \delta(\bar{\mathcal{T}})$ and the premouse $\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*}$ is ω_1 -iterable above $i_{\bar{b}}^{\bar{\mathcal{T}}^*}(\bar{\delta})$ in W using property (4) in Definition 3.1 because \mathcal{T}^* is an iteration tree on $\bar{W} \triangleleft W|\omega_1^W = N|\omega_1^N$ (using the Weak Condensation Lemma 3.3 again). Furthermore we have that \bar{W} is $2n$ -small above $\bar{\delta}$ and therefore $\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*}$ is $2n$ -small above $i_{\bar{b}}^{\bar{\mathcal{T}}^*}(\bar{\delta})$.

Moreover \mathcal{Q}_λ is Π_{2n+1}^1 -iterable above $\delta(\mathcal{T})$ in $\bar{W}[g]$ thus by Σ_{2n+1}^1 -correctness also inside W . The statement $\phi_1(\bar{N})$ yields that \mathcal{Q}_λ is $2n$ -small above $i_{\bar{b}}^{\bar{\mathcal{T}}^*}(\bar{\delta})$. We have that \bar{W} is sound by construction and thus the non-dropping iterate $\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*}$ is sound above $i_{\bar{b}}^{\bar{\mathcal{T}}^*}(\bar{\delta})$. Moreover we have that $\rho_\omega(\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*}) \leq i_{\bar{b}}^{\bar{\mathcal{T}}^*}(\bar{\delta})$. In addition \mathcal{Q}_λ is also sound above $i_{\bar{b}}^{\bar{\mathcal{T}}^*}(\bar{\delta})$ and we have that $\rho_\omega(\mathcal{Q}_\lambda) \leq \delta(\bar{\mathcal{T}}) = i_{\bar{b}}^{\bar{\mathcal{T}}^*}(\bar{\delta})$. Hence Lemma 2.12 implies that

$$\mathcal{Q}_\lambda \triangleleft \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*} \text{ or } \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*} \trianglelefteq \mathcal{Q}_\lambda.$$

So we again distinguish two different cases.

Case 1.1.1. $\mathcal{Q}_\lambda \triangleleft \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*}$.

By assumption $\bar{\delta}$ is a Woodin cardinal in \bar{W} , because N is pre- $(2n+1)$ -suitable and thus δ_N is a Woodin cardinal in W . Therefore we have by

elementarity that

$$\mathcal{M}_b^{T^*} \models "i_b^{T^*}(\bar{\delta}) \text{ is Woodin}."$$

But since \mathcal{Q}_λ is a \mathcal{Q} -structure for \mathcal{T} , we have that $\delta(\mathcal{T}) = i_b^{T^*}(\bar{\delta})$ is not definably Woodin over \mathcal{Q}_λ . This contradicts $\mathcal{Q}_\lambda \triangleleft \mathcal{M}_b^{T^*}$.

Case 1.1.2. $\mathcal{M}_b^{T^*} \trianglelefteq \mathcal{Q}_\lambda$.

In this case we have that

$$\mathcal{Q}_\lambda \cap \text{Ord} < \bar{W} \cap \text{Ord} \leq \mathcal{M}_b^{T^*} \cap \text{Ord} \leq \mathcal{Q}_\lambda \cap \text{Ord},$$

where the first inequality holds true since $\mathcal{Q}_\lambda \in \bar{W}[g]$. This contradiction finishes Case 1.1.

Case 1.2. $\delta(\bar{\mathcal{T}}) < i_b^{T^*}(\bar{\delta})$.

In this case we have that

$$\mathcal{M}_b^{T^*} \models " \delta(\bar{\mathcal{T}}) \text{ is not Woodin} ",$$

because otherwise $\mathcal{M}_b^{T^*}$ would not be $(2n+1)$ -small. This implies that $\mathcal{Q}_\lambda = \mathcal{Q}(\bar{\mathcal{T}}) \triangleleft \mathcal{M}_b^{T^*}$ and therefore we have that

$$\mathcal{Q}(\bar{\mathcal{T}}) \trianglelefteq \mathcal{M}_b^{\bar{\mathcal{T}}}.$$

Now we can again consider the statement

$$\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}})) \equiv \text{ "there is a cofinal branch } b \text{ through } \bar{\mathcal{T}} \text{ such that} \\ \mathcal{Q}(\bar{\mathcal{T}}) \trianglelefteq \mathcal{M}_b^{\bar{\mathcal{T}}} \text{ " .}$$

Again $\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}}))$ holds in the model W as witnessed by the branch \bar{b} . By an absoluteness argument as above we have that it also holds in $\bar{W}[g]$, which contradicts the fact that $\bar{\mathcal{T}}$ witnesses in $\bar{W}[g]$ that \bar{N} is not short tree iterable.

Case 2. $\bar{\mathcal{T}}$ witnesses that $\phi_2(\bar{N})$ holds in $\bar{W}[g]$.

In this case $\bar{\mathcal{T}}$ is a tree of length $\bar{\lambda} + 1$ for some ordinal $\bar{\lambda}$.

Since $\phi_2(\bar{N})$ holds true in $\bar{W}[g]$, there exists a model $\bar{\mathcal{P}} \supseteq \mathcal{M}_\lambda^{T^*} | i_{0\bar{\lambda}}^{T^*}(\bar{\delta})$, which is $2n$ -small above $i_{0\bar{\lambda}}^{T^*}(\bar{\delta})$, sound above $i_{0\bar{\lambda}}^{T^*}(\bar{\delta})$ and Π_{2n+1}^1 -iterable above $i_{0\bar{\lambda}}^{T^*}(\bar{\delta})$ in $\bar{W}[g]$. Moreover $i_{0\bar{\lambda}}^{T^*}(\bar{\delta})$ is not definably Woodin over $\bar{\mathcal{P}}$ and we have that $\rho_\omega(\bar{\mathcal{P}}) \leq i_{0\bar{\lambda}}^{T^*}(\bar{\delta})$.

Consider the comparison of $\bar{\mathcal{P}}$ with $\mathcal{M}_\lambda^{T^*}$ inside the model W . The comparison takes place above $i_{0\bar{\lambda}}^{T^*}(\bar{\delta})$ and we have that $\mathcal{M}_\lambda^{T^*}$ is $2n$ -small above $i_{0\bar{\lambda}}^{T^*}(\bar{\delta})$ because W is $2n$ -small above δ_N . The premouse $\mathcal{M}_\lambda^{T^*}$ is ω_1 -iterable above $i_{0\bar{\lambda}}^{T^*}(\bar{\delta})$ in W by the same argument we gave above in Case 1 using property (4) in Definition 3.1. Therefore the coiteration is successful using Lemma 2.12 by the following argument.

We have that $\bar{\mathcal{P}}$ is Π_{2n+1}^1 -iterable inside the model $\bar{W}[g]$ and thus by Σ_{2n+1}^1 -correctness also inside \bar{W} . The statement $\phi_2(\bar{N})$ yields that $\bar{\mathcal{P}}$ is also $2n$ -small above $i_{0\lambda}^{\bar{\mathcal{T}}^*}(\bar{\delta})$. We have that \bar{W} is sound by construction and thus the non-dropping iterate $\mathcal{M}_\lambda^{\bar{\mathcal{T}}^*}$ is sound above $i_{0\lambda}^{\bar{\mathcal{T}}^*}(\bar{\delta})$. Moreover we have that $\rho_\omega(\mathcal{M}_\lambda^{\bar{\mathcal{T}}^*}) \leq i_{0\lambda}^{\bar{\mathcal{T}}^*}(\bar{\delta})$. In addition $\bar{\mathcal{P}}$ is also sound above $i_{0\lambda}^{\bar{\mathcal{T}}^*}(\bar{\delta})$ and we have that $\rho_\omega(\bar{\mathcal{P}}) \leq i_{0\lambda}^{\bar{\mathcal{T}}^*}(\bar{\delta})$ because of $\phi_2(\bar{N})$. Hence Lemma 2.12 implies that

$$\bar{\mathcal{P}} \triangleleft \mathcal{M}_\lambda^{\bar{\mathcal{T}}^*} \text{ or } \mathcal{M}_\lambda^{\bar{\mathcal{T}}^*} \trianglelefteq \bar{\mathcal{P}}.$$

So we consider two different cases.

Case 2.1. $\bar{\mathcal{P}} \triangleleft \mathcal{M}_\lambda^{\bar{\mathcal{T}}^*}$.

By assumption $\bar{\delta}$ is a Woodin cardinal in \bar{W} , because N is pre- $(2n+1)$ -suitable and thus δ_N is a Woodin cardinal in W . Therefore we have by elementarity that

$$\mathcal{M}_\lambda^{\bar{\mathcal{T}}^*} \models \text{“}i_{0\lambda}^{\bar{\mathcal{T}}^*}(\bar{\delta}) \text{ is Woodin”}.$$

Moreover we have by the statement $\phi_2(\bar{N})$ that $i_{0\lambda}^{\bar{\mathcal{T}}^*}(\bar{\delta})$ is not definably Woodin over $\bar{\mathcal{P}}$. This is a contradiction to $\bar{\mathcal{P}} \triangleleft \mathcal{M}_\lambda^{\bar{\mathcal{T}}^*}$.

Case 2.2. $\mathcal{M}_\lambda^{\bar{\mathcal{T}}^*} \trianglelefteq \bar{\mathcal{P}}$.

In this case we have that

$$\bar{\mathcal{P}} \cap \text{Ord} < \bar{W} \cap \text{Ord} \leq \mathcal{M}_\lambda^{\bar{\mathcal{T}}^*} \cap \text{Ord} \leq \bar{\mathcal{P}} \cap \text{Ord},$$

where the first inequality holds since $\bar{\mathcal{P}} \in \bar{W}[g]$. This is a contradiction.

Therefore we proved that

$$M_{2n}(\mathcal{M}_\lambda^{\bar{\mathcal{T}}} | \delta_{\mathcal{M}_\lambda^{\bar{\mathcal{T}}}}) \models \text{“}\delta_{\mathcal{M}_\lambda^{\bar{\mathcal{T}}}} \text{ is Woodin”},$$

if $\bar{\mathcal{T}}$ is as in Case 2 above, that means if $\bar{\mathcal{T}}$ is a short iteration tree on N of length $\lambda+1$ which is non-dropping on the main branch.

This shows that there is an ordinal $\delta < \omega_1^V$ such that properties (1) and (2) in Definition 3.1 hold for the premouse $\mathcal{M}_\lambda^{\bar{\mathcal{T}}}$. That property (3) holds for $\mathcal{M}_\lambda^{\bar{\mathcal{T}}}$ follows from property (3) for the pre- $(2n+1)$ -suitable premouse N by a similar argument and property (4) follows from the corresponding property for N as well. Thus $\mathcal{M}_\lambda^{\bar{\mathcal{T}}}$ is pre- $(2n+1)$ -suitable, as desired.

Case 3. $\bar{\mathcal{T}}$ witnesses that $\phi_3(\bar{N})$ holds in $\bar{W}[g]$.

Let $\bar{\mathcal{P}} \triangleright \mathcal{M}(\bar{\mathcal{T}})$ witness that $\phi_3(\bar{N})$ holds inside $\bar{W}[g]$. We have that inside W there exists a cofinal well-founded branch \bar{b} through the iteration tree $\bar{\mathcal{T}}^*$ by property (4) in the definition of pre- $(2n+1)$ -suitability for N as above in Case 1.

Case 3.1. There is a drop along \bar{b} .

Then we have as in Case 1 that there exists a \mathcal{Q} -structure $\mathcal{Q}(\bar{\mathcal{T}}) \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$ for $\bar{\mathcal{T}}$. Consider the statement

$$\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}})) \equiv \text{“there is a cofinal branch } b \text{ through } \bar{\mathcal{T}} \text{ such that } \mathcal{Q}(\bar{\mathcal{T}}) \trianglelefteq \mathcal{M}_b^{\bar{\mathcal{T}}} \text{”}.$$

This statement $\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}}))$ is Σ_1^1 -definable from the parameters $\bar{\mathcal{T}}$ and $\mathcal{Q}(\bar{\mathcal{T}})$ and holds in the model W as witnessed by the branch \bar{b} . By an absoluteness argument as above it follows that it also holds in $\bar{W}[g]$, which contradicts the fact that $\bar{\mathcal{T}}$ witnesses in $\bar{W}[g]$ that $\phi_3(\bar{N})$ holds.

Case 3.2. There is no drop along \bar{b} .

Then we can consider the coiteration of $\bar{\mathcal{P}}$ and $\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*}$ inside the model W . We have that both premice are $2n$ -small above $\delta(\bar{\mathcal{T}})$. Moreover this coiteration takes place above $\delta(\bar{\mathcal{T}})$ since we have that $\bar{\mathcal{P}} \triangleright \mathcal{M}(\bar{\mathcal{T}})$. Therefore the coiteration is successful inside W using Lemma 2.12 by the same argument as the one we gave in Cases 1 and 2, because in W we have that $\bar{\mathcal{P}}$ is Π_{2n+1}^1 -iterable above $\delta(\bar{\mathcal{T}})$ and $\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*}$ is ω_1 -iterable above $\delta(\bar{\mathcal{T}})$ in W using property (4) in Definition 3.1. That means we have that

$$\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*} \trianglelefteq \bar{\mathcal{P}} \text{ or } \bar{\mathcal{P}} \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*}.$$

Case 3.2.1. $\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*} \trianglelefteq \bar{\mathcal{P}}$.

In this case we have that

$$\bar{\mathcal{P}} \cap \text{Ord} < \bar{W} \cap \text{Ord} \leq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*} \cap \text{Ord} \leq \bar{\mathcal{P}} \cap \text{Ord},$$

where the first inequality holds true since $\bar{\mathcal{P}} \in \bar{W}[g]$. This is a contradiction.

Case 3.2.2. $\bar{\mathcal{P}} \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*}$.

Then we have that in fact

$$\bar{\mathcal{P}} \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}},$$

because $\delta(\bar{\mathcal{T}})$ is the largest cardinal in $\bar{\mathcal{P}}$. This contradicts $\phi_3(\bar{N})$.

Therefore it follows for an iteration tree \mathcal{T} as in Case 3 that there exists a cofinal well-founded branch through \mathcal{T} and if there exists a non-dropping such branch b , then the premouse $\mathcal{M}_b^{\mathcal{T}}$ is pre- $(2n+1)$ -suitable analogous to the argument at the end of Case 2 above.

Now the argument we just gave for Case 3 shows that in this case we have that in fact

$$\mathcal{M}_b^{\mathcal{T}} = M_{2n}(\mathcal{M}(\mathcal{T})) | (\delta(\mathcal{T})^+)^{M_{2n}(\mathcal{M}(\mathcal{T}))}.$$

□

3.2. Correctness for n -suitable Premice. In the following lemmas we prove some correctness results for suitable premice in the sense of Definition 3.8. We are stating these lemmas only for the levels $(2n + 1)$ at which we proved that there exists a $(2n + 1)$ -suitable premouse (see Lemma 3.9).

Lemma 3.10. *Let $n \geq 0$ and $z \in {}^\omega\omega$. Assume that $M_{2n}^\#(x)$ exists for all $x \in {}^\omega\omega$ and let N be a $(2n + 1)$ -suitable z -premouse. Let φ be an arbitrary Σ_{2n+3}^1 -formula and let $a \in N \cap {}^\omega\omega$ be arbitrary. Then we have*

$$\varphi(a) \leftrightarrow \Vdash_{\text{Col}(\omega, \delta_N)}^N \varphi(a),$$

where δ_N as usual denotes the largest cardinal in N .

Here we again write a for the standard name \check{a} for a real $a \in N$.

Proof. Let $n \geq 0$ and $z \in {}^\omega\omega$ be arbitrary and let N be a $(2n + 1)$ -suitable z -premouse. Let $\varphi(a)$ be a Σ_{2n+3}^1 -formula for a parameter $a \in N \cap {}^\omega\omega$. That means

$$\varphi(a) \equiv \exists x \forall y \psi(x, y, a)$$

for a Σ_{2n+1}^1 -formula $\psi(x, y, a)$. We first want to prove the downward implication, that means we want to prove that if $\varphi(a)$ holds in V , then

$$\Vdash_{\text{Col}(\omega, \delta_N)}^N \varphi(a).$$

Let $x^* \in V$ be a witness for the fact that $\varphi(a)$ holds in V . That means x^* is a real such that

$$V \models \forall y \psi(x^*, y, a).$$

Use Corollary 1.8 from [Ne95] to make the real x^* generic over an iterate of N for the collapse of the image of δ_N . Since N is $(2n + 1)$ -suitable, we have enough iterability to apply Corollary 1.8 from [Ne95], so there exists a non-dropping iterate N^* of N such that N^* is $2n$ -iterable and whenever g is $\text{Col}(\omega, \delta_{N^*})$ -generic over N^* , then $x^* \in N^*[g]$. Moreover let $i : N \rightarrow N^*$ denote the corresponding iteration embedding. Since N is $(2n + 1)$ -suitable and the iteration from N to N^* is non-dropping, we have that N^* is pre- $(2n + 1)$ -suitable.

Let $g \in V$ be $\text{Col}(\omega, \delta_{N^*})$ -generic over N^* . Since we have that $N^* = M_{2n}(N^* | \delta_{N^*}) | (\delta_{N^*}^+)^{M_{2n}(N^* | \delta_{N^*})}$, it follows that g is also $\text{Col}(\omega, \delta_{N^*})$ -generic over the proper class model $M_{2n}(N^* | \delta_{N^*})$. Moreover we can construe the model $M_{2n}(N^* | \delta_{N^*})[g]$ as a y -premouse for some real y (in fact $y = z \oplus x^*$, see for example [SchSt09] for the fine structural details) which has $2n$ Woodin cardinals. This yields by Lemma 1.17 that the premouse $M_{2n}(N^* | \delta_{N^*})[g]$ is Σ_{2n+2}^1 -correct in V , because $M_{2n}(N^* | \delta_{N^*})[g]$ construed as a y -premouse is ω_1 -iterable above δ_{N^*} . Thus we have that

$$M_{2n}(N^* | \delta_{N^*})[g] \models \forall y \psi(x^*, y, a)$$

as $x^* \in M_{2n}(N^* | \delta_{N^*})[g]$. This in fact can be obtained using only Σ_{2n+1}^1 -correctness of $M_{2n}(N^* | \delta_{N^*})[g]$ and downward absoluteness.

Since the premice $M_{2n}(N^*|\delta_{N^*})[g]$ and $N^*[g]$ agree on their reals it follows that

$$N^*[g] \models \forall y \psi(x^*, y, a).$$

By homogeneity of the forcing $\text{Col}(\omega, \delta_{N^*})$ we now obtain that

$$\Vdash_{\text{Col}(\omega, \delta_{N^*})}^{N^*} \exists x \forall y \psi(x, y, a).$$

Since a is a real in N it follows by elementarity that

$$\Vdash_{\text{Col}(\omega, \delta_N)}^N \exists x \forall y \psi(x, y, a),$$

as desired.

For the upward implication let $\varphi(a) \equiv \exists x \forall y \psi(x, y, a)$ again be a Σ_{2n+3}^1 -formula for a real a in N and a Σ_{2n+1}^1 -formula $\psi(x, y, a)$ and assume that we have

$$\Vdash_{\text{Col}(\omega, \delta_N)}^N \exists x \forall y \psi(x, y, a).$$

Let g be $\text{Col}(\omega, \delta_N)$ -generic over the premouse $M_{2n}(N|\delta_N)$ and pick a real $x^* \in M_{2n}(N|\delta_N)[g]$ such that

$$M_{2n}(N|\delta_N)[g] \models \forall y \psi(x^*, y, a).$$

Since $M_{2n}(N|\delta_N)|(\delta_N^+)^{M_{2n}(N|\delta_N)} = N$ is countable in V , we can pick $g \in V$ and then get that $x^* \in V$. As above we can consider $M_{2n}(N|\delta_N)[g]$ as an ω_1 -iterable y -premouse for some real y and therefore we have by Lemma 1.17 again that

$$M_{2n}(N|\delta_N)[g] \prec_{\Sigma_{2n+2}^1} V.$$

Hence we have that

$$V \models \exists x \forall y \psi(x, y, a),$$

witnessed by the real x^* , because “ $\forall y \psi(x^*, y, a)$ ” is a Π_{2n+2}^1 -formula. \square

Lemma 3.11. *Let $n \geq 0$ and assume that $M_{2n}^\#(x)$ exists for all $x \in {}^\omega\omega$. Let N be a $(2n+1)$ -suitable z -premouse for some $z \in {}^\omega\omega$. Then $N|\delta_N$ is closed under the operation*

$$A \mapsto M_{2n}^\#(A).$$

Proof. It is enough to consider sets A of the form $N|\xi$ for some ordinal $\xi < \delta_N$ for the following reason. Let $A \in N|\delta_N$ be arbitrary. Then there exists an ordinal $\xi < \delta_N$ such that $A \in N|\xi$. Assume first that the ordinal ξ is not overlapped by an extender on the N -sequence, that means there is no extender E on the N -sequence such that $\text{crit}(E) \leq \xi < \text{lh}(E)$. We will consider the case that ξ is overlapped by an extender on the N -sequence later. Moreover assume that we already proved that

$$M_{2n}^\#(N|\xi) \triangleleft N|\delta_N.$$

Then we also have that $M_{2n}^\#(A) \in N|\delta_N$ by the following argument. Consider the model $M_{2n}(N|\xi)$. Let $L[E](A)^{M_{2n}(N|\xi)}$ denote the result of a fully backgrounded extender construction above A in the sense of [MS94] with the smallness hypothesis weakened inside the model $M_{2n}(N|\xi)$. Then

add the top measure of the active premouse $M_{2n}^\#(N|\xi)$ (intersected with $L[E](A)^{M_{2n}(N|\xi)}$) to an initial segment of $L[E](A)^{M_{2n}(N|\xi)}$ as described in Section 2 of [FNS10]. The main result in [FNS10] yields that the model we obtain from this construction is again ω_1 -iterable and not $2n$ -small. Thus it follows that the ω_1 -iterable premouse $M_{2n}^\#(A)$ exists inside $N|\delta_N$ as $M_{2n}^\#(N|\xi) \triangleleft N|\delta_N$.

So let $\xi < \delta_N$ be an ordinal and assume as above first that ξ is not overlapped by an extender on the N -sequence. Then we consider the premouse $M_{2n}^\#(N|\xi)$, which exists in V by assumption and is not $2n$ -small above ξ because ξ is countable in V . In this case we are left with showing that

$$M_{2n}^\#(N|\xi) \triangleleft N|\delta_N.$$

Let x be a real in V which codes the countable premouse $M_{2n}^\#(N|\xi)$ and $N|\delta_N$. Work inside the model $M_{2n}^\#(x)$ and coiterate $M_{2n}^\#(N|\xi)$ with $N|\delta_N$. We have that $N|\delta_N$ is short tree iterable inside $M_{2n}^\#(x)$, because for a pre- $(2n+1)$ -suitable premouse short tree iterability is a Π_{2n+2}^1 -definable statement by Lemma 3.6 and the model $M_{2n}^\#(x)$ is Σ_{2n+2}^1 -correct in V by Lemma 1.17. In fact Lemma 1.17 implies that N has an iteration strategy which is fullness preserving in the sense of Definition 3.8 inside the model $M_{2n}^\#(x)$ by the proof of Lemma 3.9.

Note that the coiteration takes place above $N|\xi$ and that $M_{2n}^\#(N|\xi)$ is ω_1 -iterable above $N|\xi$ in V by definition. Therefore Lemma 2.11 (2) implies that the comparison inside $M_{2n}^\#(x)$ cannot fail on this side of the coiteration. Say that the coiteration yields an iteration tree \mathcal{T} on $M_{2n}^\#(N|\xi)$ and an iteration tree \mathcal{U} on $N|\delta_N$.

We have that \mathcal{U} is a short tree on $N|\delta_N$, because the $M_{2n}^\#(N|\xi)$ -side of the coiteration provides \mathcal{Q} -structures. So the coiteration terminates successfully and there is an iterate M^* of $M_{2n}^\#(N|\xi)$ via \mathcal{T} and an iterate R of $N|\delta_N$ via \mathcal{U} .

Claim 1. *The $M_{2n}^\#(N|\xi)$ -side does not move in the coiteration with $N|\delta_N$.*

Proof. Assume toward a contradiction that the $M_{2n}^\#(N|\xi)$ -side moves in the coiteration. Then we have that M^* is not sound as the coiteration takes place above ξ and thus

$$M^* \supseteq R.$$

Furthermore we can consider \mathcal{U} as a tree on the premouse $N \triangleright N|\delta_N$ with final model N^* such that there is an ordinal δ^* with

$$N^*|\delta^* = R.$$

Then \mathcal{U} is a short tree of length $\lambda + 1$ for some ordinal λ and there is no drop on the main branch on the $N|\delta_N$ -side of the coiteration. So we have

that

$$\delta^* = i_{0\lambda}^{\mathcal{U}}(\delta_N),$$

where $i_{0\lambda}^{\mathcal{U}} : N \rightarrow N^*$ denotes the corresponding iteration embedding. Moreover recall that N is $(2n+1)$ -suitable and so \mathcal{U} is obtained using the iteration strategy for N which is fullness preserving for short trees which do not drop on their main branch in the sense of Definition 3.8. Therefore we have that $M_{2n}^{\#}(N^*|\delta^*)$ is not $2n$ -small above δ^* and

$$M_{2n}^{\#}(N^*|\delta^*) \models \text{“}\delta^* \text{ is Woodin”}.$$

We have for the other side of the coiteration that $\rho_{\omega}(M^*) < \delta^*$, because $\rho_{\omega}(M_{2n}^{\#}(N|\xi)) \leq \xi < \delta^*$. Let \mathcal{Q} be the least initial segment of M^* such that δ^* is not definably Woodin over \mathcal{Q} . Recall that this means that $\mathcal{Q} \trianglelefteq M^*$ is such that

$$\mathcal{Q} \models \text{“}\delta^* \text{ is Woodin”},$$

but if $\mathcal{Q} = J_{\alpha}^{M^*}$ for some $\alpha < M^* \cap \text{Ord}$ then

$$J_{\alpha+1}^{M^*} \models \text{“}\delta^* \text{ is not Woodin”},$$

and if $\mathcal{Q} = M^*$ then $\rho_{\omega}(\mathcal{Q}) < \delta^*$ or there exists an $m < \omega$ and an $r\Sigma_m$ -definable set $A \subset \delta^*$ such that there is no $\kappa < \delta^*$ such that κ is strong up to δ^* with respect to A as witnessed by extenders on the sequence of \mathcal{Q} . In the latter case we have that in particular $\rho_{\omega}(\mathcal{Q}) \leq \delta^*$. Moreover \mathcal{Q} is $2n$ -small above δ^* , because we have $\mathcal{Q} \trianglelefteq M^*$ and

$$\mathcal{Q} \models \text{“}\delta^* \text{ is Woodin”}.$$

Furthermore \mathcal{Q} is ω_1 -iterable above δ^* .

By construction we have that the premouse $M_{2n}^{\#}(N^*|\delta^*)$ is ω_1 -iterable above δ^* . Consider the coiteration of \mathcal{Q} and $M_{2n}^{\#}(N^*|\delta^*)$ inside the model $M_{2n}^{\#}(y)$, where y is a real coding the countable premice \mathcal{Q} and $M_{2n}^{\#}(N^*|\delta^*)$. Since

$$N^*|\delta^* = R \trianglelefteq M^*$$

this coiteration takes place above δ^* . We have that $\rho_{\omega}(\mathcal{Q}) \leq \delta^*$ and

$$\rho_{\omega}(M_{2n}^{\#}(N^*|\delta^*)) \leq \delta^*.$$

Moreover \mathcal{Q} and $M_{2n}^{\#}(N^*|\delta^*)$ are both sound above δ^* . Thus the comparison is successful by Lemma 2.11 and we have that

$$\mathcal{Q} \trianglelefteq M_{2n}^{\#}(N^*|\delta^*) \text{ or } M_{2n}^{\#}(N^*|\delta^*) \trianglelefteq \mathcal{Q}.$$

The premouse $M_{2n}^{\#}(N^*|\delta^*)$ is not $2n$ -small above δ^* because N^* is pre- $(2n+1)$ -suitable. Since \mathcal{Q} is $2n$ -small above δ^* we have in fact that

$$\mathcal{Q} \triangleleft M_{2n}^{\#}(N^*|\delta^*).$$

But this implies by our choice of the premouse \mathcal{Q} that δ^* is not Woodin in $M_{2n}^{\#}(N^*|\delta^*)$, which is a contradiction because by fullness preservation we have that δ^* is a Woodin cardinal in $M_{2n}^{\#}(N^*|\delta^*)$.

Therefore we have that the $M_{2^n}^\#(N|\xi)$ -side does not move in the coiteration with $N|\delta_N$. \square

Note that the proof of Claim 1 also shows that the $M_{2^n}^\#(N|\xi)$ -side cannot win the comparison with $N|\delta_N$.

Claim 2. *The $N|\delta_N$ -side does not move in the coiteration with $M_{2^n}^\#(N|\xi)$.*

Proof. Assume toward a contradiction that the $N|\delta_N$ -side moves in this coiteration. Since the coiteration takes place above ξ this means that there is an ordinal $\gamma > \xi$ such that there is an extender E_γ^N indexed at γ on the N -sequence which is used in the coiteration. In particular γ is a cardinal in the iterate R of $N|\delta_N$. This implies regardless of whether the iteration from $N|\delta_N$ to R drops or not, that γ is a cardinal in $M_{2^n}^\#(N|\xi)$ since we have that $M_{2^n}^\#(N|\xi) \triangleleft R$ using Claim 1. In fact we have, regardless of whether the iteration from $N|\delta_N$ to R drops or not, that there is a model N' such that γ is a cardinal in N' and $M_{2^n}^\#(N|\xi) \triangleleft N'$. This is a contradiction because

$$\rho_\omega(M_{2^n}^\#(N|\xi)) \leq \xi < \gamma.$$

Therefore the $N|\delta_N$ -side also does not move in the comparison. \square

By Claims 1 and 2 we finally have that

$$M_{2^n}^\#(N|\xi) \triangleleft N|\delta_N.$$

We now have to consider the case that $A \in N|\xi$ for an ordinal $\xi < \delta_N$ such that ξ is overlapped by an extender E on the N -sequence. That means there is an extender E on the N -sequence such that $\text{crit}(E) \leq \xi < \text{lh}(E)$. Let E be the least such extender, that means the index of E is minimal among all critical points of extenders on the N -sequence overlapping ξ . By the definition of a “fine extender sequence” (see Definition 2.4 in [St10]) we have that $A \in \text{Ult}(N; E)$ and that the ordinal ξ is no longer overlapped by an extender on the $\text{Ult}(N; E)$ -sequence. Let $M = \text{Ult}(N; E)$ and consider the premouse $M|\xi$, so we have in particular that $A \in M|\xi$. The same argument as above for $N|\xi$ replaced by $M|\xi$ proves that

$$M_{2^n}^\#(M|\xi) \triangleleft N|\delta_N$$

and therefore we finally have that $M_{2^n}^\#(A) \in N|\delta_N$ by repeating the argument we already gave at the beginning of this proof. \square

From Lemma 3.11 we can obtain the following lemma as a corollary.

Lemma 3.12. *Let $n \geq 0$ and $z \in {}^\omega\omega$. Assume that $M_{2^n}^\#(x)$ exists for all $x \in {}^\omega\omega$ and let N be a $(2n+1)$ -suitable z -premouse. Then N is Σ_{2n+2}^1 -correct in V for real parameters in N and we write*

$$N \prec_{\Sigma_{2n+2}^1} V.$$

Proof. Let a be a real in N . By Lemma 3.11 we have that

$$M_{2n}^\#(a) \in N.$$

Moreover we have by Lemma 1.17 that $M_{2n}^\#(a)$ is Σ_{2n+2}^1 -correct in V . Therefore it follows that N is Σ_{2n+2}^1 -correct in V . \square

Note that we could have also proven Lemma 3.12 by an argument similar to the one we gave in the proof of Lemma 3.10 but with additionally using the uniformization property as in the proof of Lemma 1.17 (1).

3.3. Outline of the Proof. Our main goal for the rest of this paper is to give a proof of the following theorem.

Theorem 3.13. *Let $n \geq 0$ and assume there is no Σ_{n+2}^1 -definable ω_1 -sequence of pairwise distinct reals. Then the following are equivalent.*

- (1) Π_n^1 determinacy and Π_{n+1}^1 determinacy,
- (2) for all $x \in {}^\omega\omega$, $M_{n-1}^\#(x)$ exists and is ω_1 -iterable, and $M_n^\#$ exists and is ω_1 -iterable,
- (3) $M_n^\#$ exists and is ω_1 -iterable.

For $n = 0$ this is due to L. Harrington (see [Ha78]) and D. A. Martin (see [Ma70]).

The results that (3) implies (2) and that (2) implies (1) for ω_1 -iterable premice $M_n^\#$ are due to the third author for odd n (unpublished) and due to Neeman for even $n > 0$ (see Theorem 2.14 in [Ne02]), building on work of Martin and Steel (see [MaSt89]). Moreover the results that (3) implies (2) and that (2) implies (1) hold without the background hypothesis that every Σ_{n+2}^1 -definable sequence of pairwise distinct reals is countable.

We will focus on the proof of the following theorem, which is the implication “(1) \Rightarrow (3)” in Theorem 3.13 and due to the third author.

Theorem 3.14. *Let $n \geq 1$ and assume there is no Σ_{n+2}^1 -definable ω_1 -sequence of pairwise distinct reals. Moreover assume that Π_n^1 determinacy and Π_{n+1}^1 determinacy hold. Then $M_n^\#$ exists and is ω_1 -iterable.*

Remark. Let $n \geq 0$. Then we say that there exists a Σ_{n+2}^1 -definable ω_1 -sequence of pairwise distinct reals iff there exists a well-order \leq^* of ordertype ω_1 for reals such that if we let $X_{\leq^*} = \text{field}(\leq^*)$, that means if we have for all $y \in {}^\omega\omega$ that

$$y \in X_{\leq^*} \Leftrightarrow \exists x (x \leq^* y \vee y \leq^* x),$$

then there exists a Σ_{n+2}^1 -definable relation R_1 such that we have for all $x, y \in {}^\omega\omega$,

$$R_1(x, y) \Leftrightarrow x, y \in X_{\leq^*} \wedge x \leq^* y,$$

and there exists a Σ_{n+2}^1 -definable relation R_2 such that we have for all $x, w \in {}^\omega\omega$,

$$R_2(x, w) \Leftrightarrow w \in \text{WO} \wedge x \text{ is the } \|w\| - \text{th element according to } \leq^*,$$

where $\text{WO} = \{w \in {}^\omega\omega \mid w \text{ codes a well-ordering on } \mathbb{N}\}$.

In fact it is not necessary to demand the existence of R_1 above as it follows from the existence of R_2 . We decided to explicitly phrase it here to be able to emphasize later what exactly is needed in a specific argument.

We will first show in Section 3.4 how boldface determinacy for a level of the projective hierarchy can be used to prove that every sequence of pairwise distinct reals which is projective at the next level of the projective hierarchy is in fact countable.

This will enable us to conclude Theorem 2.1 from Theorem 3.14. The odd levels in the inductive proof of Theorem 3.14 will finally be proven in Section 3.5 and the even levels in Section 3.6. As mentioned before the proof for the odd and the even levels of the projective hierarchy will be different because of the periodicity in the projective hierarchy in terms of uniformization and correctness.

3.4. Making use of the Right Hypothesis. In this section we will provide one ingredient for the proof of Theorem 2.1. We will see that the results from Sections 3.5 and 3.6 can actually be obtained from boldface determinacy at the right level of the projective hierarchy. The following lemma makes this precise.

Lemma 3.15. *Let $n \geq 0$. Then $\mathbf{\Pi}_{n+1}^1$ determinacy implies that every Σ_{n+2}^1 -definable sequence of pairwise distinct reals is countable.*

Proof. Again for periodicity reasons we give a completely different proof for the even and the odd levels of the projective hierarchy. We start with the even levels, for which we will give an inner model theoretic argument using the results in Sections 3.1 and 3.2.

So let $n \geq 1$ and assume that every $\mathbf{\Pi}_{2n}^1$ -definable set of reals is determined. Assume further toward a contradiction that

$$(z_\alpha \mid \alpha < \omega_1)$$

is a Σ_{2n+1}^1 -definable ω_1 -sequence of pairwise distinct reals as defined in the remark after the statement of Theorem 3.14. So in particular there are Σ_{2n+1}^1 -definable relations R_1 and R_2 as defined in the remark after the statement of Theorem 3.14. The following proof will generalize to $\Sigma_{2n+1}^1(x)$ -definable sequences of pairwise distinct reals for $x \in {}^\omega\omega$, but we will assume that $x = 0$ to simplify the notation.

By Lemmas 3.2 and 3.9 there exists a countable $(2n - 1)$ -suitable premouse N . We aim to show that $z_\alpha \in N$ for all $\alpha < \omega_1^V$ to derive a contradiction.

For this purpose we fix an arbitrary ordinal $\alpha < \omega_1^V$ and let N^* be the model which is obtained from N by iterating the least measure in N and its images $(\alpha^+)^N$ times.

Claim 1. $z_\alpha \in N^*$.

Proof. Since N is $(2n - 1)$ -suitable and thus has a fullness preserving iteration strategy for non-dropping iteration trees, the same holds for the premouse N^* , and we have by Lemma 3.10 that for any Σ_{2n+1}^1 -formula φ with a parameter $a \in N^* \cap {}^\omega\omega$,

$$\varphi(a) \leftrightarrow \Vdash_{\text{Col}(\omega, \delta_{N^*})}^{N^*} \varphi(a).$$

Pick mutual generics g and h for $\text{Col}(\omega, \alpha)$ over N^* such that

$$N^*[g] \cap N^*[h] = N^*.$$

Since $\alpha < \delta_{N^*}$ we still have by Lemma 3.10 that for any Σ_{2n+1}^1 -formula φ and any parameter $b \in N^*[g] \cap {}^\omega\omega$,

$$\varphi(b) \leftrightarrow \Vdash_{\text{Col}(\omega, \delta_{N^*})}^{N^*[g]} \varphi(b)$$

and similarly for $N^*[h]$.

In $N^*[g]$, the ordinal $\alpha < \omega_1^V$ can be coded by a real $a \in \text{WO}$ such that $\|a\| = \alpha$. This yields that $z_\alpha \in N^*[g]$, because the sequence $(z_\alpha \mid \alpha < \omega_1^V)$ is Σ_{2n+1}^1 -definable and thus z_α is a $\Sigma_{2n+1}^1(a)$ -singleton, i.e. $\{z_\alpha\}$ is Σ_{2n+1}^1 -definable from the parameter $a \in N^*[g] \cap {}^\omega\omega$ using the binary relation R_2 as defined in the remark after the statement of Theorem 3.14. Analogously we have that $z_\alpha \in N^*[h]$.

Therefore it follows that $z_\alpha \in N^*[g] \cap N^*[h] = N^*$, as desired. \square

The premice N and N^* have the same reals since the iteration from N to N^* was obtained by iterating the least measure in N and its images $(\alpha^+)^N$ times and therefore does not drop. This yields that in fact

$$z_\alpha \in N.$$

Since $\alpha < \omega_1^V$ was arbitrary and the z_α are pairwise distinct reals this is a contradiction to the fact that N is countable. This finishes the proof for the even levels of the projective hierarchy.

Note that we in fact proved an effective version of Lemma 3.15 for the even levels of the projective hierarchy. That means for any $n \geq 1$ and $x \in {}^\omega\omega$, under the hypothesis that every $\mathbf{\Pi}_{2n-1}^1$ -definable set of reals is determined and every $\mathbf{\Pi}_{2n}^1(x)$ -definable set of reals is determined, there exists no $\Sigma_{2n+1}^1(x)$ -definable ω_1 -sequence of pairwise distinct reals.

Now assume for the odd levels of the projective hierarchy that every $\mathbf{\Pi}_{2n+1}^1$ -definable set of reals is determined. The following proof for the odd levels is completely descriptive set theoretic and uses uniformization and determinacy for the Davis game. Assume toward a contradiction that there exists a Σ_{2n+2}^1 -definable ω_1 -sequence of pairwise distinct reals. That means in particular that there exists a well-order \leq^* of ordertype ω_1 for reals as in the remark after the statement of Theorem 3.14 such that if we let $X_{\leq^*} = \text{field}(\leq^*)$, i.e. if we have for all $y \in {}^\omega\omega$,

$$y \in X_{\leq^*} \Leftrightarrow \exists x (x \leq^* y \vee y \leq^* x),$$

then there exists a Σ_{2n+2}^1 -definable relation R_1 such that we have for all $x, y \in {}^\omega\omega$,

$$R_1(x, y) \Leftrightarrow x, y \in X_{\leq^*} \wedge x \leq^* y.$$

Let A be a Π_{2n+1}^1 -definable relation such that we have for all $x, y \in {}^\omega\omega$,

$$R_1(x, y) \Leftrightarrow \exists z A(z, x, y).$$

Moreover consider the relation A_2 such that for all $u \in ({}^\omega\omega)^2$ and for all $y \in {}^\omega\omega$,

$$A_2(y, u) \Leftrightarrow A((u)_0, (u)_1, y).$$

By Π_{2n+1}^1 -uniformization (see Theorem 6C.5 in [Mo09]) there exists a Π_{2n+1}^1 -definable function F which uniformizes the set A_2 , that means we have

$$(y, F(y)) \in A_2$$

for all $y \in \text{dom}(F)$, where $\text{dom}(F) = \{x \in {}^\omega\omega \mid \exists z (y, z) \in A_2\}$. We have that $X_{\leq^*} \subseteq \text{dom}(F)$. So the relation A^* defined by

$$A^*(z, x, y, u) \Leftrightarrow A(z, x, y) \wedge z = (u)_0 \wedge x = (u)_1 \wedge u = F(y)$$

is Π_{2n+1}^1 -definable. Let A^* denote the set of all tuples (z, x, y, u) such that $A^*(z, x, y, u)$ holds.

As the set A^* is uncountable and Π_{2n+1}^1 -definable, the proof of Theorem 12.11 in [Sch14] gives that the set A^* has a perfect subset, using determinacy for the Davis game (see also [Da64]). Since by definition of A^* we have that if $A^*(z, x, y, u)$ holds, the real y uniquely determines the reals u, z and x , we immediately have the following claim.

Claim 2. *The set*

$$B = \{y \in {}^\omega\omega \mid \exists z \exists x \exists u A^*(z, x, y, u)\}$$

has a perfect subset.

For $y \in B$ we have that there exists a real x such that $R_1(x, y)$ holds. So we have in particular that $y \in X_{\leq^*} = \text{field}(\leq^*)$.

Therefore there exists a continuous function $f : \mathbb{R} \rightarrow B$ such that we can consider the following order \leq on the reals. We say for two reals x and y that

$$x \leq y \Leftrightarrow f(x) \leq^* f(y).$$

Then we immediately get that the following claim holds.

Claim 3. *The order \leq is a Σ_{2n+2}^1 -definable well-order of the reals.*

Now we can obtain the following claim using Bernstein's argument.

Claim 4. *There are two disjoint Σ_{2n+2}^1 -definable sets D and D' of size \aleph_1 without perfect subsets.*

In fact we only need one of the sets D and D' in what follows.

Proof of Claim 4. We have that CH holds, i.e. we have $2^{\aleph_0} = \aleph_1$, so there are \aleph_1 -many perfect sets of reals, and let

$$(P_\alpha \mid \alpha < \omega_1)$$

enumerate all perfect sets of reals using the well-ordering of the reals \leq as defined above. Using Claim 3 we can now pick sequences $(x_\alpha \mid \alpha < \omega_1)$ and $(y_\alpha \mid \alpha < \omega_1)$ of pairwise distinct reals in a Σ^1_{2n+2} -definable way following Bernstein's argument such that for all $\alpha < \omega_1$ the real x_α is picked such that $x_\alpha \in P_\alpha$ is least according to the well-order \leq with

$$x_\alpha \notin \{y_\beta \mid \beta < \alpha\} \cup \{x_\beta \mid \beta < \alpha\}.$$

Similarly we pick $y_\alpha \in P_\alpha$ for all $\alpha < \omega_1$ such that y_α is least according to the well-order \leq with

$$y_\alpha \notin \{x_\beta \mid \beta \leq \alpha\} \cup \{y_\beta \mid \beta < \alpha\}.$$

Now let

$$D = \{x_\alpha \mid \alpha < \omega_1\},$$

and

$$D' = \{y_\alpha \mid \alpha < \omega_1\}.$$

Then it is easy to see that D and D' are disjoint and both do not contain a perfect subset. Moreover we have by construction that they are both Σ^1_{2n+2} -definable by Claim 3. \square

Let E be a Π^1_{2n+1} -definable set such that

$$x \in D \Leftrightarrow \exists y (x, y) \in E.$$

By Π^1_{2n+1} -uniformization (see Theorem 6C.5 in [Mo09]) there exists a Π^1_{2n+1} -definable partial function $C : \mathbb{R} \rightarrow \mathbb{R}$ uniformizing E , that means we have that

$$\exists y (x, y) \in E \Leftrightarrow (x, C(x)) \in E.$$

The set C' which is defined as

$$C' = \{(x, C(x)) \mid x \in D\} = C \cap E$$

is Π^1_{2n+1} -definable. As above determinacy for the Davis game implies that C' has a perfect subset $P \subset C'$, because D and thus also C' is uncountable. Since P is a perfect set and D does not contain a perfect subset, there are reals x, y and y' such that $y \neq y'$ and $(x, y) \in P$ and $(x, y') \in P$. But we have $y = C(x) = y'$. This is a contradiction. \square

3.5. $M_{2n-1}^\#(x)$ from Boldface Π_{2n}^1 Determinacy. The goal of this section is to prove Theorem 3.14 in the case that n is odd. The proof which is presented in the following argument only works if n is odd because of the periodicity in the projective hierarchy in terms of uniformization (see Theorem 6C.5 in [Mo09]). The even levels in the statement of Theorem 3.14 have to be treated differently (see Section 3.6). So we are going to prove the following theorem.

Theorem 3.16. *Let $n \geq 1$ and assume that every Π_{2n-1}^1 -definable set of reals and every Π_{2n}^1 -definable set of reals is determined. Moreover assume that there is no Σ_{2n+1}^1 -definable ω_1 -sequence of pairwise distinct reals. Then the premouse $M_{2n-1}^\#$ exists and is ω_1 -iterable.*

The proof of Theorem 3.16 uses the following Determinacy Transfer Theorem, which is due to A. S. Kechris and the third author (see [KW08]). In the version as stated below it follows from [KW08] using [Ne95] and that Theorem 3.13 holds below $2n - 1$ inductively.

Theorem 3.17 (Determinacy Transfer Theorem). *Let $n \geq 1$. Assume determinacy for every Π_{2n-1}^1 - and every Π_{2n}^1 -definable set of reals. Then we have determinacy for all $\mathcal{D}^{(2n-1)}(< \omega^2 - \Pi_1^1)$ -definable sets of reals.³*

Proof. The lightface version of Theorem 1.10 in [KW08] (see p. 369 in [KW08]) gives that for all $n \geq 1$,

$$\text{Det}(\mathcal{D}^{(2n-2)}(< \omega^2 - \Pi_1^1)) \rightarrow [\text{Det}(\Delta_{2n}^1) \leftrightarrow \text{Det}(\mathcal{D}^{(2n-1)}(< \omega^2 - \Pi_1^1))].$$

As in the addendum §5 in [KW08] we need to argue that Π_{2n-1}^1 determinacy implies $\mathcal{D}^{(2n-2)}(< \omega^2 - \Pi_1^1)$ determinacy to obtain that it implies the right-hand side of the implication, i.e. “ $\text{Det}(\Delta_{2n}^1) \leftrightarrow \text{Det}(\mathcal{D}^{(2n-1)}(< \omega^2 - \Pi_1^1))$ ”. Recall that we assume inductively that Theorem 3.13 holds for all $m < 2n - 1$. Together with Lemma 3.15 this implies that the premouse $M_{2n-2}^\#(x)$ exists and is ω_1 -iterable for all $x \in {}^\omega\omega$ from Π_{2n-1}^1 determinacy. By Theorem 2.5 in [Ne95] this yields $\mathcal{D}^{(2n-2)}(< \omega^2 - \Pi_1^1)$ determinacy. Therefore we have that

$$\text{Det}(\Pi_{2n-1}^1) \rightarrow [\text{Det}(\Delta_{2n}^1) \leftrightarrow \text{Det}(\mathcal{D}^{(2n-1)}(< \omega^2 - \Pi_1^1))].$$

We have

$$\text{Det}(\Delta_{2n}^1) \leftrightarrow \text{Det}(\Pi_{2n}^1)$$

by Theorem 5.1 in [KS85] which is due to Martin (see [Ma73]). So in particular Π_{2n-1}^1 determinacy and Π_{2n}^1 determinacy together imply that $\mathcal{D}^{(2n-1)}(< \omega^2 - \Pi_1^1)$ determinacy holds. \square

³See Section 6D in [Mo09] for a definition and some basic facts about the game quantifier “ \mathcal{D} ”. We let “ $\mathcal{D}^{(n)}$ ” denote n successive applications of the game quantifier \mathcal{D} . For the definition of the difference hierarchy and in particular of the pointclass $\alpha - \Pi_1^1$ for an ordinal $\alpha < \omega_1$ see for example Section 31 in [Ka08].

Martin proves in [Ma08] that under the assumption that $x^\#$ exists for every real x ,

$$A \in \mathcal{D}(< \omega^2 - \Pi_1^1)$$

iff there is a formula ϕ such that for all $x \in {}^\omega\omega$

$$x \in A \text{ iff } L[x] \models \phi[x, \gamma_1, \dots, \gamma_k],$$

where $\gamma_1, \dots, \gamma_k$ are Silver indiscernibles for x . In the light of this result (see also Definition 2.7 in [Ne02] for the general case) we can obtain the following corollary of the Determinacy Transfer Theorem 3.17.

Corollary 3.18. *Let $n \geq 1$. Assume that $\mathbf{\Pi}_{2n-1}^1$ determinacy and $\mathbf{\Pi}_{2n}^1$ determinacy hold. Suppose Q is a set of reals such that there is an $m < \omega$ and a formula ϕ such that for all $x \in {}^\omega\omega$*

$$x \in Q \text{ iff } M_{2n-2}(x) \models \phi(x, E, \gamma_1, \dots, \gamma_m),$$

where E is the extender sequence of $M_{2n-2}(x)$ and $\gamma_1, \dots, \gamma_m$ are the first m indiscernibles of $M_{2n-2}(x)$. Then Q is determined.

This follows from the Determinacy Transfer Theorem 3.17 as the set Q defined in Corollary 3.18 is $\mathcal{D}^{(2n-1)}(< \omega^2 - \Pi_1^1)$ -definable. Moreover note in the statement of Corollary 3.18 that $\mathbf{\Pi}_{2n-1}^1$ determinacy inductively implies that the premouse $M_{2n-2}^\#(x)$ exists for every real x .

Now we are ready to prove Theorem 3.16.

Proof of Theorem 3.16. Let $n \geq 1$. We assume inductively that Theorem 3.14 holds for $2n-2$, that means we assume that $\mathbf{\Pi}_{2n-1}^1$ determinacy implies that $M_{2n-2}^\#(x)$ exists and is ω_1 -iterable for all reals x . These even levels will be proven in Theorem 3.20. Recall that then our hypothesis implies by Lemmas 3.2 and 3.9 that there exists a $(2n-1)$ -suitable premouse.

Let $x \in {}^\omega\omega$ and consider the simultaneous comparison of all $(2n-1)$ -suitable premice N such that N is coded by a real $y_N \leq_T x$, using the iterability stated in Definition 3.8. That means analogous to a usual comparison as in Theorem 3.14 in [St10] we iterate away the least disagreement that exists between any two of the models we are considering.

These premice successfully coiterate to a common model since they are all $(2n-1)$ -suitable and call this common iterate N_x . We could have performed this simultaneous comparison inside an inner model of V of height ω_1^V which contains x and is closed under the operation $a \mapsto M_{2n-2}^\#(a)$ and therefore the resulting premouse N_x is countable in V .

Since N_x results from a successful comparison using iterability in the sense of Definition 3.8, there either exists a $(2n-1)$ -suitable premouse N and a non-dropping iteration from N to N_x via a short iteration tree or there exists a maximal iteration tree \mathcal{T} on a $(2n-1)$ -suitable premouse such that

$$N_x = M_{2n-2}(\mathcal{M}(\mathcal{T})) \mid (\delta(\mathcal{T})^+)^{M_{2n-2}(\mathcal{M}(\mathcal{T}))}.$$

Let δ_{N_x} as usual denote the largest cardinal in N_x . Then we have that in the first case N_x is a $(2n - 1)$ -suitable premouse again by fullness preservation and so in particular the model $M_{2n-2}(N_x|\delta_{N_x})$ constructed in the sense of Definition 2.9 is a well-defined proper class premouse with $2n - 1$ Woodin cardinals. In the second case we have by maximality of \mathcal{T} that

$$M_{2n-2}(N_x|\delta_{N_x}) \models \text{“}\delta(\mathcal{T}) \text{ is Woodin”},$$

with $\delta_{N_x} = \delta(\mathcal{T})$, and $M_{2n-2}(N_x|\delta_{N_x})$ is again a well-defined premouse with $2n - 1$ Woodin cardinals.

For each formula ϕ and each $m < \omega$ let Q_m^ϕ be the set of all $x \in {}^\omega\omega$ such that

$$M_{2n-2}(N_x|\delta_{N_x}) \models \phi(E, \gamma_1, \dots, \gamma_m),$$

where E is the extender sequence of $M_{2n-2}(N_x|\delta_{N_x})$ and $\gamma_1, \dots, \gamma_m$ are indiscernibles of $M_{2n-2}(N_x|\delta_{N_x})$.

Claim 1. *For all formulas ϕ and for all $m < \omega$ the set Q_m^ϕ is determined.*

Proof. We aim to reduce determinacy for the set Q_m^ϕ to determinacy for a set Q as in Corollary 3.18.

Recall that by Lemma 3.9 we have that a premouse N is $(2n - 1)$ -suitable iff it is pre- $(2n - 1)$ -suitable, that means iff it satisfies the following properties for an ordinal δ_0 .

- (1) $N \models \text{“ZFC}^- + \delta_0 \text{ is the largest cardinal”},$

$$N = M_{2n-2}(N|\delta_0) | (\delta_0^+)^{M_{2n-2}(N|\delta_0)},$$

and for every $\gamma < \delta_0$,

$$M_{2n-2}(N|\gamma) | (\gamma^+)^{M_{2n-2}(N|\gamma)} \triangleleft N,$$

- (2) $M_{2n-2}(N|\delta_0)$ is a proper class model and

$$M_{2n-2}(N|\delta_0) \models \text{“}\delta_0 \text{ is Woodin”},$$

- (3) for every $\gamma < \delta_0$, $M_{2n-2}(N|\gamma)$ is a set, or

$$M_{2n-2}(N|\gamma) \not\models \text{“}\gamma \text{ is Woodin”},$$

and

- (4) for every $\eta < \delta_0$, $M_{2n-2}(N|\delta_0) \models \text{“}N|\delta_0 \text{ is } \eta\text{-iterable”}.$

Formally the definition of pre- $(2n - 1)$ -suitability requires that in the background universe the premouse $M_{2n-2}^\#(z)$ exists for all $z \in {}^\omega\omega$. But for a premouse N which is coded by a real $y_N \leq_T x$, the model $M_{2n-2}(x)$ can compute if N is pre- n -suitable in V , i.e. if N satisfies properties (1) - (4) above in V , by considering a fully backgrounded extender construction in the sense of [MS94] above $N|\delta_0$, where δ_0 denotes the largest cardinal in N . Therefore we can make sense of these $(2n - 1)$ -suitable premice inside the model $M_{2n-2}(x)$. For the same reason the simultaneous comparison of all such $(2n - 1)$ -suitable premice which are coded by a real $y_N \leq_T x$ as

introduced above can be computed inside $M_{2n-2}(x)$ and therefore we have that $N_x \in M_{2n-2}(x)$.

Now consider the following formula ψ_ϕ , where ϕ as above is an arbitrary formula.

$\psi_\phi(x, E, \gamma_1, \dots, \gamma_m) \equiv$ “Write N_x for the result of the simultaneous comparison of all $(2n - 1)$ -suitable premice N which are coded by a real $y_N \leq_T x$ and δ_{N_x} for the largest cardinal in N_x , then $L[\bar{E}](N_x|\delta_{N_x}) \models \phi(E^*, \gamma_1, \dots, \gamma_m)$, where E^* denotes the extender sequence of $L[\bar{E}](N_x|\delta_{N_x})$, which in turn is computed from E via a fully backgrounded extender construction over $N_x|\delta_{N_x}$ as in [MS94].”

Now let E denote the extender sequence of the x -premouse $M_{2n-2}(x)$ and let $\gamma_1, \dots, \gamma_m$ denote indiscernibles of the model $L[M_{2n-2}(x)|\delta] = M_{2n-2}(x)$, where δ is the largest Woodin cardinal in $M_{2n-2}(x)$. Then we have in particular that $\gamma_1, \dots, \gamma_m$ are indiscernibles of the model $L[\bar{E}](N_x|\delta_{N_x})$, constructed via a fully backgrounded extender construction inside $M_{2n-2}(x)$ as defined in the formula $\psi_\phi(x, E, \gamma_1, \dots, \gamma_m)$ above.

Therefore we have that

$$x \in \mathcal{Q}_m^\phi \Leftrightarrow M_{2n-2}(x) \models \psi_\phi(x, E, \gamma_1, \dots, \gamma_m),$$

because the premice $L[\bar{E}](N_x|\delta_{N_x})$ as defined above and $M_{2n-2}(N_x|\delta_{N_x})$ as in the definition of the set \mathcal{Q}_m^ϕ coiterate to the same model.

Thus Corollary 3.18 implies that \mathcal{Q}_m^ϕ is determined for any formula ϕ and any $m < \omega$. \square

The sets \mathcal{Q}_m^ϕ are Turing invariant, since the premouse N_x by definition only depends on the Turing degree of x .

Let $Th^{M_{2n-2}(N_x|\delta_{N_x})}$ denote the theory of $M_{2n-2}(N_x|\delta_{N_x})$ with indiscernibles (computed in V). That means

$$Th^{M_{2n-2}(N_x|\delta_{N_x})} = \{\phi \mid M_{2n-2}(N_x|\delta_{N_x}) \models \phi(E, \gamma_1, \dots, \gamma_m), \\ m < \omega, \phi \text{ formula}\},$$

where as above E denotes the extender sequence of $M_{2n-2}(N_x|\delta_{N_x})$ and $\gamma_1, \dots, \gamma_m$ are indiscernibles of $M_{2n-2}(N_x|\delta_{N_x})$. Then we have that the theory of $M_{2n-2}(N_x|\delta_{N_x})$ stabilizes on a cone of reals x as in the following claim.

Claim 2. *There exists a real $x_0 \geq_T x$ such that for all reals $y \geq_T x_0$,*

$$Th^{M_{2n-2}(N_{x_0}|\delta_{N_{x_0}})} = Th^{M_{2n-2}(N_y|\delta_{N_y})}.$$

Proof. By Claim 1 the set Q_m^ϕ is determined and as argued above it is also Turing invariant for all formulas ϕ and all $m < \omega$. That means the set Q_m^ϕ either contains a cone of reals or is completely disjoint from a cone of reals. For each formula ϕ and each natural number m let $x_m^\phi \in {}^\omega\omega$ be such that either $y \in Q_m^\phi$ for all $y \geq_T x_m^\phi$ or else $y \notin Q_m^\phi$ for all $y \geq_T x_m^\phi$. Let

$$x_0 = \bigoplus \{x_m^\phi \mid \phi \text{ formula}, m < \omega\}.$$

Then we have by construction for all $y \geq_T x_0$ that

$$Th^{M_{2n-2}(N_{x_0}|\delta_{N_{x_0}})} = Th^{M_{2n-2}(N_y|\delta_{N_y})},$$

as desired. \square

Let $x_0 \in {}^\omega\omega$ be as in Claim 2. We want to show that the unique theory $T = Th^{M_{2n-2}(N_x|\delta_{N_x})}$ of $M_{2n-2}(N_x|\delta_{N_x})$ with indiscernibles as defined above for $x \geq_T x_0$ in fact gives a candidate for the theory of the premouse $M_{2n-1}^\#$ in V to conclude that $M_{2n-1}^\#$ exists and is ω_1 -iterable in V . By coding a formula ϕ by its unique Gödel number $\ulcorner \phi \urcorner$ we can code the theory T by a real x_T .

Fix a real z such that $z \geq_T x_0 \oplus x_T$. Moreover we can pick the real z such that it in addition codes a $(2n-1)$ -suitable premouse by Lemmas 3.2 and 3.9.

Using the existence of the premouse $M_{2n-2}^\#(x)$ in V for every real x , we can as described below close under the operation

$$a \mapsto M_{2n-2}^\#(a)$$

to construct a transitive model M_z from z . Moreover we will define an order of construction for elements of the model M_z along the way.

The fact that the model M_z will be closed under $a \mapsto M_{2n-2}^\#(x)$ directly yields that M_z is Σ_{2n}^1 -correct (see Claim 4).

In the construction of the model M_z we aim to construct models by expanding the usual definition of Gödel's constructible universe L by adding additional elements at the successor steps of the construction. Therefore recall the following definition which is used in the definition of Jensen's J -hierarchy for L (see §1 and §2 in [Je72]).

Definition 3.19. *Let X be an arbitrary set. Then $\text{TC}(X)$ denotes the transitive closure of X and $\text{rud}(X)$ denotes the closure of $\text{TC}(X) \cup \{\text{TC}(X)\}$ under rudimentary functions (see for example Definition 1.1 in [SchZe10] for the definition of a rudimentary function).*

We construct a sequence of models $(W_\alpha \mid \alpha \leq \omega_1^V)$ and in particular the model $M_z = W_{\omega_1^V}$ level-by-level in a construction of length ω_1^V , starting from z and taking unions at limit steps of the construction. So we let $W_0 = \{z\}$.

Successor steps: At a successor level $\alpha + 1$ of the construction we close the previous model W_α under the operation $a \mapsto M_{2n-2}^\#(a)$ before closing under rudimentary functions. More precisely assume that we already constructed W_α and let $a \in W_\alpha$ be arbitrary. Then the a -premouse $M_{2n-2}^\#(a)$ exists in V because a is countable in V and we inductively assume that Theorem 3.13 holds for $2n - 2$. Let \mathcal{M} be a countable a -premouse in V with the following properties.

- (i) \mathcal{M} is $(2n - 1)$ -small, but not $(2n - 2)$ -small,
- (ii) all proper initial segments of \mathcal{M} are $(2n - 2)$ -small,
- (iii) \mathcal{M} is a -sound and $\rho_\omega(\mathcal{M}) = a$, and
- (iv) \mathcal{M} is Π_{2n}^1 -iterable.

These properties uniquely determine the a -premouse $M_{2n-2}^\#(a)$ in V . We let $W_{\alpha+1}$ be the model obtained by taking the closure under rudimentary functions of W_α together with all such a -premise \mathcal{M} as above for all $a \in W_\alpha$.

Order of construction: For an a -premouse \mathcal{M}_a and a b -premouse \mathcal{M}_b satisfying properties (i) – (iv) for $a, b \in W_\alpha$, we say that \mathcal{M}_a is defined before \mathcal{M}_b if a is defined before b in the order of construction for elements of W_α , which exists inductively. For elements added by the closure under rudimentary functions we define the order of construction analogous to the order of construction for L .

Limit steps: At a limit step of the construction we let

$$W_\lambda = \bigcup_{\alpha < \lambda} W_\alpha$$

for all limit ordinals $\lambda < \omega_1^V$ and we finally let

$$M_z = W_{\omega_1^V} = \bigcup_{\alpha < \omega_1^V} W_\alpha.$$

Order of construction: The order of construction at the limit steps is defined analogous to the order of construction for L (see Lemma 5.26 in [Sch14]).

Now we get that M_z is a model of ZFC from the background hypothesis that there is no Σ_{2n+1}^1 -definable ω_1 -sequence of pairwise distinct reals as in the following claim.

Claim 3. $M_z \models \text{ZFC}$.

Proof. Assume not. Then the power set axiom has to fail. So let γ be a countable ordinal such that

$$\mathcal{P}(\gamma) \cap M_z \notin M_z.$$

This yields that the set $\mathcal{P}(\gamma) \cap M_z$ has size \aleph_1 .

Let $W_\gamma = M_z \upharpoonright \gamma$ be the γ -th level in the construction of M_z . Then we can fix a real a in V which codes the countable set W_γ .

If it exists, we let A_ξ for $\gamma < \xi < \omega_1^V$ be the smallest subset of γ in

$$M_z|(\xi + 1) \setminus M_z|\xi$$

according to the order of construction. Moreover we let X be the set of all ξ with $\gamma < \xi < \omega_1^V$ such that A_ξ exists. Then X is cofinal in ω_1^V .

Finally we let a_ξ again be a real coding the set A_ξ relative to the code a for W_γ . For $\xi \in X$ we have that $A_\xi \in \mathcal{P}(\gamma) \cap M_z$ and thus $A_\xi \subseteq W_\gamma$, so the canonical code a_ξ for A_ξ relative to a exists.

Now consider the following ω_1^V -sequence of reals

$$A = (a_\xi \in {}^\omega\omega \mid \xi \in X).$$

We have that a real y codes an element of the model $M_z|(\xi + 1) \setminus M_z|\xi$ for some $\xi < \omega_1^V$ iff there is a sequence of countable models $(W_\beta \mid \beta \leq \xi + 1)$ and an element $Y \in W_{\xi+1}$ such that

- (1) $W_0 = \{z\}$,
- (2) $W_{\beta+1}$ is constructed from W_β as described in the construction above for all $\beta \leq \xi$,
- (3) $W_\lambda = \bigcup_{\beta < \lambda} W_\beta$ for all limit ordinals $\lambda \leq \xi$,
- (4) y does not code an element of W_ξ , and
- (5) y codes Y .

This states that the real y codes an element of the model $M_z|(\xi + 1) \setminus M_z|\xi$ as the construction of the levels of the model M_z is defined in a unique way. Now we have to argue that this statement is definable enough for our purposes.

For the successor levels of the construction we have that the properties (i) – (iv) as in the construction are Π_{2n}^1 -definable uniformly in any code for the countable premouse \mathcal{M} .

So this shows that the statement “ a_ξ codes the smallest subset of γ in $M_z|(\xi + 1) \setminus M_z|\xi$ ” is Σ_{2n+1}^1 -definable uniformly in a , z and any code for ξ and γ .

Therefore the ω_1 -sequence A as defined above is Σ_{2n+1}^1 -definable uniformly in a and z in the sense of the remark after the statement of Theorem 3.14. Thus A contradicts the assumption that every Σ_{2n+1}^1 -definable sequence of pairwise distinct reals is countable. \square

Moreover we have the following claim.

Claim 4. *The resulting model M_z has the following properties.*

- (1) $M_z \cap \text{Ord} = \omega_1^V$, $z \in M_z$,
- (2) $M_z \prec_{\Sigma_{2n}^1} V$,
- (3) M_z is closed under the operation

$$a \mapsto M_{2n-2}^\#(a),$$

and moreover $M_{2n-2}^\#(a)$ is ω_1 -iterable in M_z for all $a \in M_z$.

Proof. Property (1) immediately follows from the construction.

Proof of (3), first part: Let $a \in M_z$ be arbitrary. Then we have that $a \in W_\alpha$ for some ordinal $\alpha < \omega_1^V$. Recall that by our assumptions the ω_1 -iterable a -premouse $M_{2n-2}^\#(a)$ exists in V as a is countable in V . Moreover it satisfies properties (i) – (iv) from the construction, i.e.

- (i) $M_{2n-2}^\#(a)$ is $(2n-1)$ -small, but not $(2n-2)$ -small,
- (ii) all proper initial segments of $M_{2n-2}^\#(a)$ are $(2n-2)$ -small,
- (iii) $M_{2n-2}^\#(a)$ is a -sound and $\rho_\omega(M_{2n-2}^\#(a)) = a$, and
- (iv) $M_{2n-2}^\#(a)$ is Π_{2n}^1 -iterable.

Therefore the a -premouse $\mathcal{M} = M_{2n-2}^\#(a)$, which is ω_1 -iterable in V , has been added to the model M_z at some successor level of the construction and thus M_z is closed under the operation $a \mapsto M_{2n-2}^\#(a)$.

Proof of (2): The fact that the model M_z is closed under the operation $a \mapsto M_{2n-2}^\#(a)$ immediately implies property (2), because the a -premouse $M_{2n-2}^\#(a)$ is Σ_{2n}^1 -correct in V by Lemma 1.17.

Proof of (3), second part: We now want to show that the a -premouse \mathcal{M} as above, which has been added to the model M_z at some successor level of the construction, is ω_1 -iterable inside M_z via the iteration strategy Σ which is guided by \mathcal{Q} -structures (see Definition 2.5).

Let \mathcal{T} be an iteration tree of length λ on the a -premouse \mathcal{M} for some limit ordinal $\lambda < \omega_1$ in M_z such that \mathcal{T} is according to the iteration strategy Σ . Then \mathcal{T} is guided by \mathcal{Q} -structures which are $(2n-2)$ -small above $\delta(\mathcal{T} \upharpoonright \gamma)$, ω_1 -iterable above $\delta(\mathcal{T} \upharpoonright \gamma)$ and thus also Π_{2n-1}^1 -iterable above $\delta(\mathcal{T} \upharpoonright \gamma)$ in M_z for all limit ordinals $\gamma \leq \lambda$.

By (2) we have that M_z is Σ_{2n}^1 -correct in V for real parameters in M_z . Therefore it follows that the \mathcal{Q} -structures $\mathcal{Q}(\mathcal{T} \upharpoonright \gamma)$ which are $(2n-2)$ -small above $\delta(\mathcal{T} \upharpoonright \gamma)$ and guiding the iteration tree \mathcal{T} in M_z are Π_{2n-1}^1 -iterable above $\delta(\mathcal{T} \upharpoonright \gamma)$ in V for all limit ordinals $\gamma \leq \lambda$. Since $M_{2n-2}^\#(x)$ exists in V for all $x \in {}^\omega\omega$ by our assumptions and in particular $M_{2n-2}^\#(a)$ exists in V , we have by Lemma 2.13 that these \mathcal{Q} -structures also witness that \mathcal{T} is according to the \mathcal{Q} -structure iteration strategy Σ in V . Therefore there exists a unique cofinal well-founded branch b through \mathcal{T} in V such that we have $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$. By an absoluteness argument as given several times before (see for example the proof of Lemma 2.11), it follows that the unique cofinal well-founded branch b through \mathcal{T} in V for which $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$ holds, is also contained in M_z since we have that $\mathcal{T}, \mathcal{Q}(\mathcal{T}) \in M_z$.

Therefore $\mathcal{M} = M_{2n-2}^\#(a)$ exists and is ω_1 -iterable in M_z via the iteration strategy Σ . \square

Now we aim to show that a K^c -construction in M_z reaches $M_{2n-1}^\#$, meaning that the premouse $(K^c)^{M_z}$ is not $(2n-1)$ -small. Here and in what follows we consider a K^c -construction in the sense of [MSch04] as this construction does not assume any large cardinals in the background model.

Claim 5. $(K^c)^{M_z}$ is not $(2n-1)$ -small.

Proof. We work inside the model M_z and distinguish numerous different cases. Moreover we assume toward a contradiction that $(K^c)^{M_z}$ is $(2n-1)$ -small.

First we show the following subclaim.

Subclaim 1. *Let δ be a cutpoint in $(K^c)^{M_z}$. If $(K^c)^{M_z}$ does not have a Woodin cardinal above δ , then $(K^c)^{M_z}$ is fully iterable above δ in M_z .*

We allow $\delta = \omega$, i.e. the case that $(K^c)^{M_z}$ has no Woodin cardinals, in the statement of Subclaim 1.

Proof of Subclaim 1. $(K^c)^{M_z}$ is iterated via the \mathcal{Q} -structure iteration strategy Σ (see Definition 2.5). That means for an iteration tree \mathcal{U} of limit length on $(K^c)^{M_z}$ we let

$$\Sigma(\mathcal{U}) = b \text{ iff } \mathcal{Q}(b, \mathcal{U}) = \mathcal{Q}(\mathcal{U}) \text{ and } \mathcal{Q}(\mathcal{U}) \trianglelefteq M_{2n-2}^\#(\mathcal{M}(\mathcal{U})),$$

since $(K^c)^{M_z}$ is assumed to be $(2n-1)$ -small.

It is enough to show that $(K^c)^{M_z}$ is ω_1 -iterable above δ inside M_z , because then an absoluteness argument as the one we gave in the proof of Lemma 2.11 yields that $(K^c)^{M_z}$ is fully iterable above δ inside M_z as the iteration strategy for $(K^c)^{M_z}$ is guided by \mathcal{Q} -structures.

Assume toward a contradiction that $(K^c)^{M_z}$ is not ω_1 -iterable above δ in M_z . Then there exists an iteration tree \mathcal{T} on $(K^c)^{M_z}$ above δ of limit length $< \omega_1$ inside M_z such that there exists no \mathcal{Q} -structure $\mathcal{Q}(\mathcal{T})$ for \mathcal{T} with $\mathcal{Q}(\mathcal{T}) \trianglelefteq M_{2n-2}^\#(\mathcal{M}(\mathcal{T}))$ and hence

$$M_{2n-2}^\#(\mathcal{M}(\mathcal{T})) \models \text{“}\delta(\mathcal{T}) \text{ is Woodin”}.$$

The premouse $M_{2n-2}^\#(\mathcal{M}(\mathcal{T}))$, constructed in the sense of Definition 2.9, exists in M_z and is not $(2n-2)$ -small above $\delta(\mathcal{T})$, because otherwise it would already provide a \mathcal{Q} -structure for \mathcal{T} .

Let \bar{M} be the Mostowski collapse of a countable substructure of M_z containing the iteration tree \mathcal{T} . That means we choose a large enough natural number m and let \bar{M} , X and σ be such that

$$\bar{M} \xrightarrow{\sigma} X \prec_{\Sigma_m} M_z,$$

where

$$\sigma : \bar{M} \rightarrow M_z$$

denotes the uncollapse map such that we have a model \bar{K} in \bar{M} with $\sigma(\bar{K}|\gamma) = (K^c)^{M_z}|\sigma(\gamma)$ for every ordinal $\gamma < \bar{M} \cap \text{Ord}$, and we have an

iteration tree $\bar{\mathcal{T}}$ on \bar{K} above an ordinal $\bar{\delta}$ in \bar{M} with $\sigma(\bar{\mathcal{T}}) = \mathcal{T}$ and $\sigma(\bar{\delta}) = \delta$. Moreover we have that

$$M_{2n-2}^\#(\mathcal{M}(\bar{\mathcal{T}})) \models \text{“}\delta(\bar{\mathcal{T}}) \text{ is Woodin”}.$$

By the iterability proof of Chapter 9 in [St96] (adapted as in [MSch04]) applied inside M_z , there exists a cofinal well-founded branch b through the iteration tree $\bar{\mathcal{T}}$ on \bar{K} in M_z such that we have a final model $\mathcal{M}_b^{\bar{\mathcal{T}}}$.

Consider the coiteration of $\mathcal{M}_b^{\bar{\mathcal{T}}}$ with $M_{2n-2}^\#(\mathcal{M}(\bar{\mathcal{T}}))$ and note that it takes place above $\delta(\bar{\mathcal{T}})$. Since $M_{2n-2}^\#(\mathcal{M}(\bar{\mathcal{T}}))$ is ω_1 -iterable above $\delta(\bar{\mathcal{T}})$ inside M_z and $\mathcal{M}_b^{\bar{\mathcal{T}}}$ is iterable in M_z by the iterability proof of Chapter 9 in [St96] (adapted as in [MSch04]) applied inside the model M_z , the coiteration is successful using Lemma 2.11 (2).

If there is no drop along the branch b , then $\mathcal{M}_b^{\bar{\mathcal{T}}}$ cannot lose the coiteration, because otherwise there exists a non-dropping iterate R^* of $\mathcal{M}_b^{\bar{\mathcal{T}}}$ and an iterate \mathcal{M}^* of $M_{2n-2}^\#(\mathcal{M}(\bar{\mathcal{T}}))$ such that $R^* \trianglelefteq \mathcal{M}^*$. But we have that there is no Woodin cardinal in $\mathcal{M}_b^{\bar{\mathcal{T}}}$ above $\bar{\delta}$ by elementarity and at the same time we have that $\bar{\delta} < \delta(\bar{\mathcal{T}})$ and

$$M_{2n-2}^\#(\mathcal{M}(\bar{\mathcal{T}})) \models \text{“}\delta(\bar{\mathcal{T}}) \text{ is Woodin”}.$$

If there is a drop along b , then $\mathcal{M}_b^{\bar{\mathcal{T}}}$ also has to win the coiteration, because we have $\rho_\omega(\mathcal{M}_b^{\bar{\mathcal{T}}}) < \delta(\bar{\mathcal{T}})$ and $\rho_\omega(M_{2n-2}^\#(\mathcal{M}(\bar{\mathcal{T}}))) = \delta(\bar{\mathcal{T}})$.

That means in both cases there is an iterate R^* of $\mathcal{M}_b^{\bar{\mathcal{T}}}$ and a non-dropping iterate \mathcal{M}^* of $M_{2n-2}^\#(\mathcal{M}(\bar{\mathcal{T}}))$ such that $\mathcal{M}^* \trianglelefteq R^*$. We have that \mathcal{M}^* is not $(2n-1)$ -small, because $M_{2n-2}^\#(\mathcal{M}(\bar{\mathcal{T}}))$ is not $(2n-1)$ -small as argued above and the iteration from $M_{2n-2}^\#(\mathcal{M}(\bar{\mathcal{T}}))$ to \mathcal{M}^* is non-dropping. Therefore it follows that R^* is not $(2n-1)$ -small and thus $\mathcal{M}_b^{\bar{\mathcal{T}}}$ is not $(2n-1)$ -small. By the iterability proof of Chapter 9 in [St96] (adapted as in [MSch04]) applied inside M_z we can re-embed the model $\mathcal{M}_b^{\bar{\mathcal{T}}}$ into a model of the $(K^c)^{M_z}$ -construction. This yields that $(K^c)^{M_z}$ is not $(2n-1)$ -small, contradicting our assumption that it is $(2n-1)$ -small. \square

Now we distinguish the following cases.

Case 1. Assume that

$$(K^c)^{M_z} \models \text{“there is a Woodin cardinal”}.$$

Then we can assume that there is a largest Woodin cardinal in $(K^c)^{M_z}$, because otherwise $(K^c)^{M_z}$ has infinitely many Woodin cardinals and is therefore not $(2n-1)$ -small. So let δ denote the largest Woodin cardinal in $(K^c)^{M_z}$.

Consider the premouse $\mathcal{M} = M_{2n-2}^\#((K^c)^{M_z}|\delta)$ in the sense of Definition 2.9 and note that \mathcal{M} exists inside M_z by Claim 4. Now try to coiterate $(K^c)^{M_z}$ with \mathcal{M} inside M_z .

Since the comparison takes place above δ and the premouse

$$\mathcal{M} = M_{2n-2}^\#((K^c)^{M_z}|\delta)$$

is ω_1 -iterable above δ , the coiteration is successful, using Lemma 2.11 (2) and Subclaim 1 since the iteration strategies for the premice $(K^c)^{M_z}$ and $M_{2n-2}^\#((K^c)^{M_z}|\delta)$ above δ are guided by \mathcal{Q} -structures which are $(2n-2)$ -small above $\delta(\mathcal{T})$ for any iteration tree \mathcal{T} of limit length (see proof of Claim 4).

So there is an iterate R of $(K^c)^{M_z}$ and an iterate \mathcal{M}^* of \mathcal{M} such that the coiteration terminates with $R \trianglelefteq \mathcal{M}^*$ or $\mathcal{M}^* \trianglelefteq R$. By universality of the model $(K^c)^{M_z}$ inside M_z (see Section 3 in [MSch04]) we have that $\mathcal{M}^* \trianglelefteq R$ and that there is no drop on the \mathcal{M} -side of the coiteration. This implies that the premouse $\mathcal{M} = M_{2n-2}^\#((K^c)^{M_z}|\delta)$ in the sense of Definition 2.9 is not $(2n-2)$ -small above δ , as otherwise we have that \mathcal{M} is not fully sound and since $\mathcal{M}^* \trianglelefteq R$ this yields a contradiction because of soundness.

Therefore \mathcal{M} is not $(2n-1)$ -small and thus \mathcal{M}^* is also not $(2n-1)$ -small. That means R and thereby $(K^c)^{M_z}$ is not $(2n-1)$ -small, which is the desired contradiction.

This finishes the proof of Claim 5 in the case that there is a Woodin cardinal in $(K^c)^{M_z}$.

Case 2. Assume that

$$(K^c)^{M_z} \not\models \text{“there is a Woodin cardinal”}.$$

Recall that by our choice of the real z , there exists a premouse N (coded by the real z) in M_z which is $(2n-1)$ -suitable in V and try to coiterate $(K^c)^{M_z}$ with N inside M_z . We again iterate $(K^c)^{M_z}$ via the \mathcal{Q} -structure iteration strategy Σ as in Subclaim 1. So using Subclaim 1 the coiteration cannot fail on the $(K^c)^{M_z}$ -side.

We have that the premouse N is $(2n-1)$ -suitable in V . Since the statement “ N is pre- $(2n-1)$ -suitable” is Π_{2n}^1 -definable uniformly in any code for N , it follows that N is pre- $(2n-1)$ -suitable inside M_z , because M_z is Σ_{2n}^1 -correct in V and closed under the operation $a \mapsto M_{2n-2}^\#(a)$. Therefore Lemma 3.9 implies that N is $(2n-1)$ -suitable inside M_z . In particular N has an iteration strategy which is fullness preserving for non-dropping iteration trees.

So we have that the coiteration of $(K^c)^{M_z}$ with N is successful. Let \mathcal{T} and \mathcal{U} be the resulting trees on $(K^c)^{M_z}$ and N of length $\lambda+1$ for some ordinal λ . It follows that $(K^c)^{M_z}$ wins the comparison by universality of the premouse $(K^c)^{M_z}$ inside M_z (see Section 3 in [MSch04]). Therefore we have that there

exists an iterate $R = \mathcal{M}_\lambda^\mathcal{T}$ of $(K^c)^{M_z}$ and an iterate $N^* = \mathcal{M}_\lambda^\mathcal{U}$ of N such that $N^* \leq R$. Moreover there is no drop on the main branch on the N -side of the coiteration.

Since N is $(2n - 1)$ -suitable we have that $M_{2n-2}^\#(N^*|\delta_{N^*})$ is a premouse with $2n - 1$ Woodin cardinals, which is ω_1 -iterable above δ_{N^*} and not $(2n - 2)$ -small above δ_{N^*} . So we can consider the coiteration of R with $M_{2n-2}^\#(N^*|\delta_{N^*})$. This coiteration is successful using Lemma 2.11 (2) because we have

$$N^*|\delta_{N^*} = R|\delta_{N^*}$$

and R and $M_{2n-2}^\#(N^*|\delta_{N^*})$ are both iterable above δ_{N^*} , using Subclaim 1 for the iterate R of $(K^c)^{M_z}$. If there is no drop on the main branch in \mathcal{T} , then $R = \mathcal{M}_\lambda^\mathcal{T}$ wins the comparison by universality of $(K^c)^{M_z}$ inside M_z (again see Section 3 in [MSch04]). If there is a drop on the main branch in \mathcal{T} , then R also wins the comparison, because in this case we have that

$$\rho_\omega(R) < \delta_{N^*} \text{ and } \rho_\omega(M_{2n-2}^\#(N^*|\delta_{N^*})) = \delta_{N^*}.$$

Therefore we have that there exists an iterate R^* of R and an iterate M^* of $M_{2n-2}^\#(N^*|\delta_{N^*})$ such that we have $M^* \leq R^*$ in both cases and the iteration from $M_{2n-2}^\#(N^*|\delta_{N^*})$ to M^* is non-dropping. Since $M_{2n-2}^\#(N^*|\delta_{N^*})$ is not $(2n - 1)$ -small, we have that M^* and therefore R^* is not $(2n - 1)$ -small. Thus R is not $(2n - 1)$ -small. But R is an iterate of $(K^c)^{M_z}$ and therefore this implies that $(K^c)^{M_z}$ is not $(2n - 1)$ -small, contradicting our assumption.

This finishes the proof of Claim 5. \square

Claim 5 now implies that $M_{2n-1}^\#$ exists in M_z as the minimal ω_1 -iterable premouse which is not $(2n - 1)$ -small.

Work inside the model M_z and let $x \in M_z$ be a real which codes x_0 , the theory T and the premouse $(M_{2n-1}^\#)^{M_z}$. Let

$$N^* = (M_{2n-1}^\# | (\delta_0^+)^{M_{2n-1}^\#})^{M_z}$$

denote the *suitable initial segment* of $(M_{2n-1}^\#)^{M_z}$, where δ_0 denotes the least Woodin cardinal in $(M_{2n-1}^\#)^{M_z}$. Then we have in particular that N^* is a $(2n - 1)$ -suitable premouse (in M_z).

Recall that by Lemma 3.9 we have that N^* is a $(2n - 1)$ -suitable premouse iff it is pre- $(2n - 1)$ -suitable, that means iff it satisfies properties (1) – (4) listed in the proof of Claim 1. Therefore the statement “ N^* is a $(2n - 1)$ -suitable premouse” is Π_{2n}^1 -definable uniformly in any code for N^* . Hence by Σ_{2n}^1 -correctness of M_z in V it follows that N^* is also a $(2n - 1)$ -suitable premouse in V .

Recall that N_x denotes the common iterate of all $(2n - 1)$ -suitable premice N which are coded by a real y_N recursive in x . Since M_z is Σ_{2n}^1 -correct in V it follows that the premouse N_x is the same computed in M_z or in V . This yields by correctness of M_z in V again that the premouse

$M_{2n-2}^\#(N_x|\delta_{N_x})$ is the same computed in M_z or in V by the following argument. The premouse $(M_{2n-2}^\#(N_x|\delta_{N_x}))^{M_z}$ is Π_{2n}^1 -iterable above δ_{N_x} in M_z and by Σ_{2n}^1 -correctness of M_z also in V . Therefore we can successfully coiterate the premice $(M_{2n-2}^\#(N_x|\delta_{N_x}))^{M_z}$ and $(M_{2n-2}^\#(N_x|\delta_{N_x}))^V$ inside V by Lemma 2.12 since the latter premouse is ω_1 -iterable in V above δ_{N_x} and the comparison takes place above δ_{N_x} . It follows that in fact

$$(M_{2n-2}^\#(N_x|\delta_{N_x}))^{M_z} = (M_{2n-2}^\#(N_x|\delta_{N_x}))^V.$$

As mentioned before, we have that $N^* = (M_{2n-1}^\#|\delta_0^+)^{M_{2n-1}^\#}^{M_z}$ is a $(2n-1)$ -suitable premouse in M_z and in V and since x codes the premouse $(M_{2n-1}^\#)^{M_z}$, the premouse N^* is coded by a real recursive in x .

Consider the comparison of N^* with $N_x = (N_x)^{M_z}$ inside an inner model of M_z of height $\omega_1^{M_z}$ which is closed under the operation $a \mapsto M_{2n-2}^\#(a)$. The premouse $(M_{2n-1}^\#)^{M_z}$ is ω_1 -iterable in M_z and therefore it follows that N^* is ω_1 -iterable in M_z . Thus arguments as in the proof of Lemma 2.11 yield that N_x is in fact a non-dropping iterate of N^* , because N^* is one of the models giving rise to N_x .

The same argument shows that N_x does not move in the comparison with $(M_{2n-1}^\#)^{M_z}$. So in fact there is a non-dropping iterate M of $(M_{2n-1}^\#)^{M_z}$ below $(\delta_0^+)^{(M_{2n-1}^\#)^{M_z}}$ such that $N_x \trianglelefteq M$. Since the iteration from $(M_{2n-1}^\#)^{M_z}$ to M is fullness preserving in the sense of Definition 3.8 and it takes place below $(\delta_0^+)^{(M_{2n-1}^\#)^{M_z}}$ in M_z , we have that in fact

$$M = M_{2n-2}^\#(N_x|\delta_{N_x}),$$

because $(M_{2n-1}^\#)^{M_z} = M_{2n-2}^\#(N^*|\delta_0)$. Therefore we have that $M_{2n-2}^\#(N_x|\delta_{N_x})$ is a non-dropping iterate of $M_{2n-2}^\#(N^*|\delta_0)$ and hence

$$Th^{M_{2n-2}(N_x|\delta_{N_x})} = Th^{M_{2n-2}(N^*|\delta_0)}.$$

Recall that by Claim 2 we picked the real x_0 such that $Th^{M_{2n-2}(N_x|\delta_{N_x})}$ and thus $Th^{M_{2n-2}(N^*|\delta_0)}$ is constant for all $x \geq_T x_0$. This now implies that the theory of $(M_{2n-1}^\#)^{M_z}$ is constant for all $z \geq_T x_0 \oplus x_T$, where x_T is as above a real coding the theory T .

Thus if we now work in V and let $N = (M_{2n-1}^\#)^{M_z}$ for $z \geq_T x_0 \oplus x_T$, then we have

$$(M_{2n-1}^\#)^{M_z} = N = (M_{2n-1}^\#)^{M_y}$$

for all $y \geq_T x_0 \oplus x_T$. We aim to show that

$$(M_{2n-1}^\#)^V = N,$$

so in particular that $(M_{2n-1}^\#)^V$ exists.

For this reason we inductively show that the premouse N is ω_1 -iterable in V via the \mathcal{Q} -structure iteration strategy Σ (see Definition 2.5). So assume that

\mathcal{T} is an iteration tree via Σ of limit length $< \omega_1$ on N (in V). So we have that the branch b through the iteration tree $\mathcal{T} \upharpoonright \lambda$ is given by \mathcal{Q} -structures, i.e. $\mathcal{Q}(b, \mathcal{T} \upharpoonright \lambda) = \mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$, for every limit ordinal $\lambda < \text{lh}(\mathcal{T})$.

Pick $z \in {}^\omega\omega$ with $z \geq_T x_0 \oplus x_T$ such that $\mathcal{T} \in M_z$ and $\text{lh}(\mathcal{T}) < \omega_1^{M_z}$. Since \mathcal{T} is an iteration tree on $N = (M_{2n-1}^\#)^{M_z}$ according to the iteration strategy Σ in V , we have that for all limit ordinals $\lambda < \text{lh}(\mathcal{T})$ the \mathcal{Q} -structure $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ exists in V and is $(2n-1)$ -small above $\delta(\mathcal{T} \upharpoonright \lambda)$. In fact $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ is not more complicated than the least active premouse which is not $(2n-2)$ -small above $\delta(\mathcal{T} \upharpoonright \lambda)$. So in this case we have that Π_{2n}^1 -iterability for these \mathcal{Q} -structures is enough to determine a unique cofinal well-founded branch b through \mathcal{T} . Since M_z is Σ_{2n}^1 -correct in V it follows that \mathcal{T} is also according to the \mathcal{Q} -structure iteration strategy Σ inside M_z . Moreover recall that

$$(M_{2n-1}^\#)^{M_z} = N.$$

Therefore there exists a cofinal well-founded branch b through \mathcal{T} in M_z . As above this branch is determined by \mathcal{Q} -structures $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ which are ω_1 -iterable above $\delta(\mathcal{T} \upharpoonright \lambda)$ and therefore Π_{2n}^1 -iterable above $\delta(\mathcal{T} \upharpoonright \lambda)$ in M_z for all limit ordinals $\lambda \leq \text{lh}(\mathcal{T})$. That means in particular that we have $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$. Moreover $\mathcal{Q}(\mathcal{T})$ is also Π_{2n}^1 -iterable in V and therefore it follows that b is the unique cofinal well-founded branch determined by these \mathcal{Q} -structures in V as well. So N is ω_1 -iterable in V via the \mathcal{Q} -structure iteration strategy Σ .

Thus we now finally have that

$$V \models \text{“}M_{2n-1}^\# \text{ exists and is } \omega_1\text{-iterable”}.$$

This finishes the proof of Theorem 3.14 for odd $n < \omega$. \square

3.6. $M_{2n}^\#(x)$ from Boldface Π_{2n+1}^1 Determinacy. In this section we will finish the proof of Theorem 2.1 by proving Theorem 3.20 which will yield Theorem 3.14 and finally Theorem 2.1 for even levels n in the projective hierarchy (using Lemma 3.15). Therefore together with the previous section we will have Theorem 2.1 for arbitrary levels n .

Theorem 3.20. *Let $n \geq 1$ and assume that there is no Σ_{2n+2}^1 -definable ω_1 -sequence of pairwise distinct reals. Moreover assume that Π_{2n}^1 determinacy and Π_{2n+1}^1 determinacy hold. Then $M_{2n}^\#$ exists and is ω_1 -iterable.*

Recall that Lemma 3.15 gives that Π_{2n+1}^1 determinacy suffices to prove that every Σ_{2n+2}^1 -definable sequence of pairwise distinct reals is countable.

In order to prove Theorem 3.20 we are considering slightly different premice than before. The main advantage of these models is that they only contain partial extenders on their extender sequence and therefore behave nicer in some arguments to follow. The premice we want to consider are defined as follows.

Definition 3.21. Let A be an arbitrary countable transitive swo'd⁴ set. With $Lp^n(A)$ we denote the unique model of height ω_1^V called lower part model above A which is given by the following recursive definition. We start with $N_0 = A$. Assume that we already constructed N_α . Then we let $N_{\alpha+1}$ be the model theoretic union of all countable A -premise $M \supseteq N_\alpha$ such that

- (1) M is n -small above $N_\alpha \cap \text{Ord}$,
- (2) $\rho_\omega(M) \leq N_\alpha \cap \text{Ord}$,
- (3) M is sound above $N_\alpha \cap \text{Ord}$,
- (4) $N_\alpha \cap \text{Ord}$ is a cutpoint of M , and
- (5) M is ω_1 -iterable above $N_\alpha \cap \text{Ord}$.

For the limit step let λ be a limit ordinal and assume that we already defined N_α for all $\alpha < \lambda$. Then we let N_λ be the model theoretic union of all A -premise N_α for $\alpha < \lambda$.

We finally let $Lp^n(A) = N_{\omega_1^V}$.

A special case of this is $Lp^n(x)$ for $x \in {}^\omega\omega$.

Remark. We have that every lower part model as defined above does not contain total extenders.

We will state the following lemmas for lower part models constructed above reals x instead of swo'd sets A as this will be our main application. But they also hold (with the same proofs) if we replace x with a countable transitive swo'd set A as in Definition 3.21.

Lemma 3.22. Let $n \geq 1$ and assume that Π_{2n}^1 determinacy holds. Moreover let $x \in {}^\omega\omega$ be arbitrary. Then the lower part model above x , $Lp^{2n-1}(x)$, is well-defined, that means $Lp^{2n-1}(x)$ is an x -premouse.

Proof. Recall that we assume inductively that Theorem 3.16 holds. That means Π_{2n}^1 determinacy implies that $M_{2n-1}^\#(x)$ exists for all $x \in {}^\omega\omega$. Therefore we have by Lemma 2.11 that whenever M and M' are two x -premise extending some x -premouse N_α (as in Definition 3.21) and satisfying properties (1)-(5) in Definition 3.21 for some $x \in {}^\omega\omega$, then we have that in fact $M \trianglelefteq M'$ or $M' \trianglelefteq M$. Therefore $Lp^{2n-1}(x)$ is a well-defined x -premouse. \square

Lemma 3.23. Let $n \geq 1$ and assume that Π_{2n}^1 determinacy holds. Moreover let $x \in {}^\omega\omega$ be arbitrary. Let M denote the ω_1^V -th iterate of $M_{2n-1}^\#(x)$ by its least measure and its images. Then

$$M|_{\omega_1^V} = Lp^{2n-1}(x).$$

Proof. Let $x \in {}^\omega\omega$ and let $(N_\alpha \mid \alpha \leq \omega_1^V)$ be the sequence of models from the definition of $Lp^{2n-1}(x)$ (see Definition 3.21). We aim to show inductively

⁴i.e. self-wellordered as for example defined in Definition 3.1 in [SchT].

that

$$N_\alpha \trianglelefteq M$$

for all $\alpha < \omega_1^V$, where M denotes the ω_1^V -th iterate of $M_{2n-1}^\#(x)$ by its least measure and its images as in the statement of Lemma 3.23. Fix an $\alpha < \omega_1^V$ and assume inductively that

$$N_\beta \trianglelefteq M$$

for all $\beta \leq \alpha$. Let z be a real which codes the countable premice $N_{\alpha+1}$ and $M_{2n-1}^\#(x)$.

Since $M_{2n-1}^\#(x)$ is ω_1 -iterable in V and has no definable Woodin cardinals, we have by Lemma 2.11 (2) that it is $(\omega_1 + 1)$ -iterable inside the model $M_{2n-1}(z)$. Therefore we have in particular that M is $(\omega_1 + 1)$ -iterable inside $M_{2n-1}(z)$.

Recall that $N_{\alpha+1}$ is by definition the model theoretic union of all countable x -premise $N \triangleright N_\alpha$ such that

- (1) N is $(2n - 1)$ -small above $N_\alpha \cap \text{Ord}$,
- (2) $\rho_\omega(N) \leq N_\alpha \cap \text{Ord}$,
- (3) N is sound above $N_\alpha \cap \text{Ord}$,
- (4) $N_\alpha \cap \text{Ord}$ is a cutpoint of N , and
- (5) N is ω_1 -iterable above $N_\alpha \cap \text{Ord}$.

In particular Lemma 2.11 (2) implies that all these x -premise $N \triangleright N_\alpha$ satisfying properties (1) – (5) are $(\omega_1 + 1)$ -iterable above $N_\alpha \cap \text{Ord}$ inside the model $M_{2n-1}(z)$ since they have no definable Woodin cardinals above $N_\alpha \cap \text{Ord}$. In particular $N_{\alpha+1}$ is well-defined as in Lemma 3.22 and it follows that $N_{\alpha+1}$ is $(\omega_1 + 1)$ -iterable above $N_\alpha \cap \text{Ord}$ inside $M_{2n-1}(z)$.

Hence we can consider the comparison of the x -premise M and $N_{\alpha+1}$ inside the model $M_{2n-1}(z)$. This comparison is successful because by our inductive hypothesis it takes place above $N_\alpha \cap \text{Ord}$. Now we distinguish the following cases.

Case 1. Both sides of the comparison move.

In this case both sides of the comparison have to drop since we have that $\rho_\omega(N) \leq N_\alpha \cap \text{Ord}$ for all x -premise N occurring in the definition of $N_{\alpha+1}$, $N_{\alpha+1} \cap \text{Ord} < \omega_1^V$ and M only has partial extenders on its sequence below ω_1^V . Let \mathcal{T} and \mathcal{U} be iteration trees of length $\lambda + 1$ for some ordinal λ on M and $N_{\alpha+1}$ respectively resulting from the comparison. Moreover let E_β and F_γ be the first extenders used in the coiteration after the last drop along $[0, \lambda]_{\mathcal{T}}$ and $[0, \lambda]_{\mathcal{U}}$ respectively. Then by the proof of the Comparison Lemma (see Theorem 3.11 in [St10]) the extenders E_β and F_γ are compatible. Again by the proof of the Comparison Lemma this is a contradiction, so only one side of the coiteration can move.

Case 2. Only the M -side of the comparison moves.

As above we have in this case that the M -side drops. So there is an iterate M^* of M such that $N_{\alpha+1}$ is an initial segment of M^* . Let E_β for some ordinal β be the first extender used on the M -side in the coiteration of M with $N_{\alpha+1}$. In particular E_β is an extender indexed on the M -sequence above $N_\alpha \cap \text{Ord}$ and below ω_1^V . Since E_β has to be a partial extender, there exists a $(2n-1)$ -small sound countable x -premouse $N \triangleleft M$ such that E_β is a total extender on the N -sequence and $\rho_m(N) \leq \text{crit}(E_\beta)$ for some natural number m . Moreover we have that N is ω_1 -iterable by the iterability of M . Furthermore we have that $\text{crit}(E_\beta)$ is a cardinal in $N_{\alpha+1}$, because the extender E_β is used in the coiteration and we have that

$$N_{\alpha+1} \trianglelefteq M^*.$$

Therefore the premouse N is contained in $N_{\alpha+1}$ by the definition of a lower part model. But this contradicts the assumption that E_β was used in the coiteration, because then we have that there is no disagreement between M and $N_{\alpha+1}$ at E_β .

Case 3. Only the $N_{\alpha+1}$ -side of the comparison moves.

In this case there is an iterate N^* of $N_{\alpha+1}$ above $N_\alpha \cap \text{Ord}$ such that the iteration from $N_{\alpha+1}$ to N^* drops and we have that

$$M \trianglelefteq N^*.$$

Recall that M denotes the ω_1^V -th iterate of $M_{2n-1}^\#(x)$ by its least measure and its images and is therefore in particular not $(2n-1)$ -small above ω_1^V . But then the same holds for N^* and thus it follows that $N_{\alpha+1}$ is not $(2n-1)$ -small above $N_\alpha \cap \text{Ord}$, which is a contradiction, because $N_{\alpha+1}$ is the model theoretic union of premice which are $(2n-1)$ -small above $N_\alpha \cap \text{Ord}$.

This proves that

$$N_{\alpha+1} \trianglelefteq M$$

for all $\alpha < \omega_1^V$ and since $Lp^{2n-1}(x) \cap \text{Ord} = N_{\omega_1^V} \cap \text{Ord} = \omega_1^V$ we finally have that

$$Lp^{2n-1}(x) = M|_{\omega_1^V}.$$

□

Remark. Lemma 3.23 also implies that the lower part model $Lp^{2n-1}(x)$ is closed under the operation $a \mapsto M_{2n-2}^\#(a)$ by the proof of Lemma 2.11 (1) as this holds for $M|_{\omega_1^V}$, where M again denotes the ω_1^V -th iterate of $M_{2n-1}^\#(x)$ by its least measure and its images.

Using this representation of lower part models we can also prove the following lemma.

Lemma 3.24. *Let $n \geq 1$ and assume that $\mathbf{\Pi}_{2n}^1$ determinacy holds. Let $x, y \in {}^\omega\omega$ be such that $x \in Lp^{2n-1}(y)$. Then we have that*

$$Lp^{2n-1}(x) \subseteq Lp^{2n-1}(y)$$

and in fact $Lp^{2n-1}(x)$ is definable inside $Lp^{2n-1}(y)$ from the parameter x plus possibly finitely many ordinal parameters. If in addition we have that $y \leq_T x$, then

$$Lp^{2n-1}(x) = (Lp^{2n-1}(x))^{Lp^{2n-1}(y)}.$$

Here $(Lp^{2n-1}(x))^{Lp^{2n-1}(y)}$ denotes the model of height ω_1^V which is constructed analogous to Definition 3.21, but with models M which are ω_1 -iterable above $N_\alpha \cap \text{Ord}$ inside $Lp^{2n-1}(y)$ instead of inside V .

Proof. Let $x, y \in \omega\omega$ be such that $x \in Lp^{2n-1}(y)$. We first prove that

$$Lp^{2n-1}(x) \subseteq Lp^{2n-1}(y).$$

By Lemma 3.23 we have that $Lp^{2n-1}(x) = M(x)|\omega_1^V$, where $M(x)$ denotes the ω_1^V -th iterate of $M_{2n-1}^\#(x)$ by its least measure and its images. Moreover we have that $Lp^{2n-1}(y) = M(y)|\omega_1^V$, where $M(y)$ denotes the ω_1^V -th iterate of $M_{2n-1}^\#(y)$ by its least measure and its images. Let $M^*(y)$ denote the result of iterating the top measure of $M(y)$ out of the universe.

Consider the result of a fully backgrounded extender construction (in the sense of [MS94] but with the smallness hypothesis weakened) above x inside the model $M^*(y)$, which we denote by

$$L[E](x)^{M^*(y)}.$$

Moreover let $M_x^\#$ denote the model obtained from $L[E](x)^{M^*(y)}$ by adding the top measure (intersected with $L[E](x)^{M^*(y)}$) of the active premouse $M(y)$ to an initial segment of $L[E](x)^{M^*(y)}$ as in Section 2 of [FNS10].

We can successfully compare the active x -premise $M_x^\#$ and $M(x)$ inside the model $M_{2n-1}(z)$, where z is a real coding $M_{2n-1}^\#(y)$ and $M_{2n-1}^\#(x)$, by an argument using Lemma 2.11 as in the proof of Lemma 2.18. Therefore it follows that $M_x^\# = M(x)$ and thus we have that in fact

$$L[E](x)^{M^*(y)}|\omega_1^V = M_x^\#|\omega_1^V = M(x)|\omega_1^V.$$

Since $L[E](x)^{M^*(y)} \subseteq M^*(y)$ it follows that

$$Lp^{2n-1}(x) = M(x)|\omega_1^V = L[E](x)^{M^*(y)}|\omega_1^V \subseteq M^*(y)|\omega_1^V = Lp^{2n-1}(y).$$

Now we prove that $Lp^{2n-1}(x)$ is definable inside $Lp^{2n-1}(y)$ from the parameter x plus possibly finitely many ordinal parameters. Let $(N_\alpha \mid \alpha \leq \omega_1^V)$ be the sequence of models from the definition of $Lp^{2n-1}(x)$ in V (see Definition 3.21) and let $(N_\alpha^{Lp^{2n-1}(y)} \mid \alpha \leq \omega_1^V)$ denote the corresponding sequence of models from the definition of $(Lp^{2n-1}(x))^{Lp^{2n-1}(y)}$. Let $\alpha < \omega_1^V$ be such that $N_\alpha = N_\alpha^{Lp^{2n-1}(y)}$ and let $N \supseteq N_\alpha$ be an x -premouse which satisfies properties (1) – (5) in Definition 3.21 in V . So in particular we have that $N \in Lp^{2n-1}(x)$ and N is $(2n-1)$ -small above $N_\alpha \cap \text{Ord}$. As argued above we have that $N \in Lp^{2n-1}(y)$ and we first want to show that as N is ω_1 -iterable

above $N_\alpha \cap \text{Ord}$ in V it follows that N is ω_1 -iterable above $N_\alpha \cap \text{Ord}$ inside $Lp^{2n-1}(y)$.

So assume that N is ω_1 -iterable above $N_\alpha \cap \text{Ord}$ in V and recall that the model $Lp^{2n-1}(y)$ is closed under the operation $a \mapsto M_{2n-2}^\#(a)$. Since N is $(2n-1)$ -small above $N_\alpha \cap \text{Ord}$ and has no definable Woodin cardinal above $N_\alpha \cap \text{Ord}$, we have that for an iteration tree \mathcal{T} on N of length $< \omega_1$ in $Lp^{2n-1}(y)$ above $N_\alpha \cap \text{Ord}$ the iteration strategy Σ is guided by \mathcal{Q} -structures which are $(2n-2)$ -small above $\delta(\mathcal{T} \upharpoonright \lambda)$ for every limit ordinal $\lambda \leq \text{lh}(\mathcal{T})$. Therefore the \mathcal{Q} -structures for \mathcal{T} are contained in the model $Lp^{2n-1}(y)$ and we have that N is ω_1 -iterable inside $Lp^{2n-1}(y)$ above $N_\alpha \cap \text{Ord}$ if we argue analogous to the proof of Lemma 2.11 (2).

Assume now that $Lp^{2n-1}(x) \neq (Lp^{2n-1}(x))^{Lp^{2n-1}(y)}$, as otherwise the definability of $Lp^{2n-1}(x)$ inside $Lp^{2n-1}(y)$ is trivial. That means there is an ordinal $\alpha < \omega_1^V$ such that $N_\alpha = N_\alpha^{Lp^{2n-1}(y)}$ and there exists a premouse $N \triangleright N_\alpha$ which satisfies properties (1) – (5) in the definition of $(Lp^{2n-1}(x))^{Lp^{2n-1}(y)}$, so in particular N is ω_1 -iterable above $N_\alpha \cap \text{Ord}$ inside $Lp^{2n-1}(y)$, but N is not ω_1 -iterable above $N_\alpha \cap \text{Ord}$ in V .

Recall the model $M^*(y)$ from the first part of this proof. We have that $Lp^{2n-1}(x) = L[E](x)^{M^*(y)}|_{\omega_1^V}$ and N are both sufficiently iterable in $M^*(y)$, so we can consider the coiteration of $Lp^{2n-1}(x)$ and N inside $M^*(y)$ and distinguish the following cases.

Case 1. Both sides of the comparison move.

In this case both sides of the comparison have to drop since we have that $\rho_\omega(N) \leq N_\alpha \cap \text{Ord}$ and $Lp^{2n-1}(x)$ only has partial extenders on its sequence. As in Case 1 in the proof of Lemma 3.23 this yields a contradiction.

Case 2. Only the $Lp^{2n-1}(x)$ -side of the comparison moves.

Then the $Lp^{2n-1}(x)$ -side drops and we have that there is an iterate M of $Lp^{2n-1}(x)$ such that $N \leq M$. But this would imply that N is ω_1 -iterable in V , contradicting our choice of N .

Case 3. Only the N -side of the comparison moves.

In this case there exists an iterate N^* of N such that $Lp^{2n-1}(x) \leq N^*$. In fact the iteration from N to N^* only uses measures of Mitchell order 0 as the iteration cannot leave any total measures behind. Since there are only finitely many drops along the main branch of an iteration tree, this implies that the whole iteration from N to N^* can be defined over $L[N]$ from N and a finite sequence of ordinals. As $N \triangleleft (Lp^{2n-1}(x))^{Lp^{2n-1}(y)}$ we therefore get that $Lp^{2n-1}(x)$ is definable inside $Lp^{2n-1}(y)$ from the parameter x plus finitely many ordinal parameters.

Now we prove the “moreover” part of Lemma 3.24, so assume that we in addition have that $y \leq_T x$. By the argument we just gave,

$$Lp^{2n-1}(x) \subsetneq (Lp^{2n-1}(x))^{Lp^{2n-1}(y)} \subseteq Lp^{2n-1}(y).$$

This contradicts the following claim.

Claim 1. *For reals x, y such that $x \in Lp^{2n-1}(y)$ and $y \leq_T x$ we have that*

$$Lp^{2n-1}(x) = Lp^{2n-1}(y).$$

Proof. As $x \in Lp^{2n-1}(y)$ we have that $x \in M(y)$ and thus $x \in M_{2n-1}^\#(y)$, because $M(y)$ is obtained from $M_{2n-1}^\#(y)$ by iterating its least measure and its images.

As in the proof of Lemma 2.18 we can consider the premice $L[E](x)^{M_{2n-1}(y)}$ and $L[E](y)^{L[E](x)^{M_{2n-1}(y)}}$ and if we let κ denote the least measurable cardinal in $M_{2n-1}(y)$, then we get as in the proof of Lemma 2.18 that

$$V_\kappa^{M_{2n-1}(y)} = V_\kappa^{L[E](x)^{M_{2n-1}(y)}}.$$

Moreover if we let $(L[E](x)^{M_{2n-1}(y)})^\#$ denote the premouse obtained by adding the restriction of the top extender of $M_{2n-1}^\#(y)$ to an initial segment of the premouse $L[E](x)^{M_{2n-1}(y)}$ as in [FNS10], then another comparison argument analogous to Lemma 2.18 yields that $M_{2n-1}^\#(x) = (L[E](x)^{M_{2n-1}(y)})^\#$. Therefore we have that

$$V_\kappa^{M_{2n-1}(y)} = V_\kappa^{L[E](x)^{M_{2n-1}(y)}} = V_\kappa^{M_{2n-1}(x)}.$$

This implies that $Lp^{2n-1}(x) = Lp^{2n-1}(y)$, as desired. \square

\square

In general it need not be the case that $Lp^{2n-1}(x) = (Lp^{2n-1}(x))^{Lp^{2n-1}(y)}$ as the following counterexample shows.

Lemma 3.25. *Assume that Π_2^1 determinacy holds. Then there exists a real x and a premouse M such that M is ω_1 -iterable in $Lp^1(x)$, but M is not ω_1 -iterable in V .*

Proof sketch. Consider the premouse M_1 and let δ denote the Woodin cardinal in M_1 . Force with the countable stationary tower (see [La04]) over M_1 and let

$$j : M_1 \rightarrow N$$

be the corresponding generic elementary embedding with $\text{crit}(j) = \omega_1^{M_1}$ and $j(\omega_1^{M_1}) = \delta$. Moreover let M denote the shortest initial segment of N such that $M_1 \cap \mathbb{R} \subset M$ and $\rho_\omega(M) = \omega$. Then we can identify the premouse M with a canonical real m coding M .

$M_1|\delta$ thinks that all of its own initial segments are fully iterable, so that $N|j(\delta)$ thinks that M is fully iterable. Moreover, a short argument shows

that N is short tree iterable in V , i.e. N is iterable in V according to iteration trees \mathcal{T} on N such that $\mathcal{Q}(\mathcal{T}) \leq L[\mathcal{M}(\mathcal{T})]$. This is proven using the fact that M_1 and thus N is a model of the statement “I am short tree iterable” together with an absoluteness argument similar to the one given for example in the proof of Lemma 3.9.

Now consider the coiteration of $M_1(m)$ and N (construed as an m -premouse) in V and let \mathcal{T} and \mathcal{U} denote the resulting trees on $M_1(m)$ and N respectively, where the cofinal branches at limit steps in \mathcal{U} are given by the short tree iteration strategy for N . So there exists an iterate M^* of $M_1(m)$ and a pseudo-normal-iterate (see Definition 3.13 in [StW16]) N^* of N such that $M^* = N^*$. Here we either have that N^* is the last model of \mathcal{U} , or we have that the iteration tree \mathcal{U} is maximal and $N^* = L[\mathcal{M}(\mathcal{U})]$. Moreover there is no drop on the main branch in \mathcal{T} , which is leading from $M_1(m)$ to M^* , and therefore the iteration embedding $i : M_1(m) \rightarrow M^*$ exists. As M is fully iterable inside $N|j(\delta)$, M will also be fully iterable inside N^* , cut off at its Woodin cardinal (to conclude this, we don’t need that there is a map from N to N^*). Hence M is fully iterable inside M^* , cut off at its Woodin cardinal, and then by elementarity it follows that M is ω_1 -iterable inside $M_1(m)$ and thus

$$Lp^1(m) \models \text{“}M \text{ is } \omega_1\text{-iterable”}.$$

But the premouse M is not ω_1 -iterable in V . \square

We also have a version of Lemma 2.3 for lower part models $Lp^{2n-1}(x)$ as follows.

Lemma 3.26. *Let $n \geq 1$ and assume that Π_{2n}^1 determinacy and Π_{2n+1}^1 determinacy hold. Then there exists a real x such that we have for all reals $y \geq_T x$ that*

$$Lp^{2n-1}(y) \models \text{OD-determinacy}.$$

Proof. Recall that we assume inductively that Π_{2n}^1 determinacy implies that $M_{2n-1}^\#(x)$ exists for all $x \in {}^\omega\omega$. Then Lemma 3.26 follows from Lemma 2.3 by the following argument. For $x \in {}^\omega\omega$ we have that

$$M_{2n-1}(x)|\delta_x \models \text{OD-determinacy}$$

implies that

$$M_{2n-1}(x)|\kappa \models \text{OD-determinacy},$$

where κ is the least measurable cardinal in $M_{2n-1}(x)$, because whenever a set of reals A is ordinal definable in $M_{2n-1}(x)|\kappa$, then it is also ordinal definable in $M_{2n-1}(x)|\delta_x$, since $M_{2n-1}(x)|\kappa$ and $M_{2n-1}(x)|\delta_x$ have the same sets of reals. This yields by elementarity that

$$M|\omega_1^V \models \text{OD-determinacy},$$

where M denotes the ω_1^V -th iterate of $M_{2n-1}(x)$ by its least ω measure and its images. By Lemma 3.23 we have that $M|\omega_1^V = Lp^{2n-1}(x)$, so it follows

that

$$Lp^{2n-1}(x) \models \text{OD-determinacy}.$$

□

This immediately yields that we have the following variant of Theorem 2.23.

Theorem 3.27. *Let $n \geq 1$ and assume that Π_{2n}^1 determinacy and Π_{2n+1}^1 determinacy hold. Then there exists a real x such that we have for all reals $y \geq_T x$ that*

$$\text{HOD}^{Lp^{2n-1}(y)} \models \text{“}\omega_2^{Lp^{2n-1}(y)} \text{ is inaccessible”}.$$

Now we can turn to the proof of Theorem 3.20, which is going to yield Theorem 3.14 in the case that n is even.

Proof of Theorem 3.20. We start with constructing a ZFC model M_x of height ω_1^V for some $x \in {}^\omega\omega$ such that M_x is Σ_{2n+2}^1 -correct in V for real parameters in M_x and closed under the operation $a \mapsto M_{2n-1}^\#(a)$. To prove that this construction yields a model of ZFC, we are as before going to use the hypothesis that there is no Σ_{2n+2}^1 -definable ω_1 -sequence of pairwise distinct reals. The construction will be similar to the construction we gave in Section 3.5, but here we have to manually ensure that M_x is Σ_{2n+2}^1 -correct in V .

Fix an arbitrary $x \in {}^\omega\omega$ and construct a sequence of models $(W_\alpha \mid \alpha \leq \omega_1^V)$. So the model $M_x = W_{\omega_1^V}$ is build level-by-level in a construction of length ω_1^V . We are starting from $W_0 = \{x\}$ and are taking unions at limit steps of the construction. At an odd successor level $\alpha + 1$ we will close the model W_α under Skolem functions for Σ_{2n+2}^1 -formulas. At the same time we will use the even successor levels $\alpha + 2$ to ensure that M_x will be closed under the operation $a \mapsto M_{2n-1}^\#(a)$. As before the order of construction for elements of the model M_x can be defined along the way.

Before we are describing this construction in more detail, we fix a Π_{2n+1}^1 -definable set U which is universal for the pointclass Π_{2n+1}^1 . Pick the set U such that we have $U_{\ulcorner\varphi\urcorner a} = A_{\varphi,a}$ for every Π_{2n+1}^1 -formula φ and every $a \in {}^\omega\omega$, where $\ulcorner\varphi\urcorner$ denotes the Gödel number of the formula φ and

$$A_{\varphi,a} = \{x \mid \varphi(x, a)\}.$$

Then the uniformization property (see Theorem 6C.5 in [Mo09]) yields that there exists a Π_{2n+1}^1 -definable function F uniformizing the universal set U . So we have for all $z \in \text{dom}(F)$ that

$$(z, F(z)) \in U,$$

where $\text{dom}(F) = \{z \mid \exists y (z, y) \in U\}$.

Odd successor steps: For the odd successor steps of the construction assume now that we already constructed the model W_α such that $\alpha + 1$ is odd and that there exists a Π_{2n+1}^1 -formula φ with a real parameter a from

W_α such that $\exists x\varphi(x, a)$ holds in V but not in the model W_α . In this case we aim to add a real $x_{\varphi, a}$ constructed as described below to $W_{\alpha+1}$ such that $\varphi(x_{\varphi, a}, a)$ holds. This real will witness that $\exists x\varphi(x, a)$ holds true inside $W_{\alpha+1}$.

We aim to build these levels of M_x in a Σ_{2n+2}^1 -definable way, so we choose reals $x_{\varphi, a}$ carefully. Therefore we add $F(z)$ for all $z \in \text{dom}(F) \cap W_\alpha$ to the current model W_α . We will see in Claims 1 and 2 that this procedure adds reals $x_{\varphi, a}$ as above in a Π_{2n+1}^1 -definable way to the model M_x .

So let φ_F be a Π_{2n+1}^1 -formula such that for all $x, y \in {}^\omega\omega$,

$$F(x) = y \text{ iff } \varphi_F(x, y).$$

Then we let

$$W_{\alpha+1} = \text{rud}(W_\alpha \cup \{y \in {}^\omega\omega \mid \exists x \in W_\alpha \cap {}^\omega\omega \varphi_F(x, y)\}).$$

Order of construction: We inductively define the order of construction for elements of $W_{\alpha+1}$ as follows. First we say for $F(x) \neq F(x')$ with $x, x' \in \text{dom}(F) \cap W_\alpha$ that $F(x)$ is constructed before $F(x')$ iff x is constructed before x' in the order of construction for elements of W_α where x and x' are the minimal (according to the order of construction in W_α) reals y and y' in $\text{dom}(F) \cap W_\alpha$ such that $F(y) = F(x)$ and $F(y') = F(x')$. Then we define the order of construction for elements added by the closure under rudimentary functions analogous to the order of construction for L .

Even successor steps: At an even successor level $\alpha+2$ of the construction we close the previous model $W_{\alpha+1}$ under the operation $a \mapsto M_{2n-1}^\#(a)$ before closing under rudimentary functions. Assume that we already constructed $W_{\alpha+1}$ and let $a \in W_{\alpha+1}$ be arbitrary. The a -premouse $M_{2n-1}^\#(a)$ exists in V because we as usual inductively assume that Theorem 3.13 holds for $2n-1$. Analogous to the proof of Theorem 3.16 let \mathcal{M} be a countable a -premouse in V with the following properties.

- (i) \mathcal{M} is $2n$ -small, but not $(2n-1)$ -small,
- (ii) all proper initial segments of \mathcal{M} are $(2n-1)$ -small,
- (iii) \mathcal{M} is a -sound and $\rho_\omega(\mathcal{M}) = a$, and
- (iv) \mathcal{M} is Π_{2n+1}^1 -iterable.

We have that these properties (i) – (iv) uniquely determine the a -premouse $M_{2n-1}^\#(a)$ in V and we add such a -premouse \mathcal{M} for all $a \in W_{\alpha+1}$ to the model $W_{\alpha+2}$ before closing under rudimentary functions as in the usual construction of L .

Order of construction: For an a -premouse \mathcal{M}_a and a b -premouse \mathcal{M}_b satisfying properties (i) – (iv) for $a, b \in W_{\alpha+1}$, we say that \mathcal{M}_a is defined before \mathcal{M}_b if a is defined before b in the order of construction for elements of $W_{\alpha+1}$, which exists inductively. For elements added by the closure under rudimentary functions we define the order of construction analogous to the order of construction for L .

Limit steps: Finally we let

$$W_\lambda = \bigcup_{\alpha < \lambda} W_\alpha$$

for limit ordinals $\lambda < \omega_1^V$ and

$$M_x = W_{\omega_1^V} = \bigcup_{\alpha < \omega_1^V} W_\alpha.$$

Order of construction: The order of construction at the limit steps is defined analogous to the order of construction for L .

As in Section 3.5 we are now able to show that this model M_x satisfies ZFC, using the background hypothesis that every Σ_{2n+2}^1 -definable sequence of pairwise distinct reals is countable.

Claim 1. $M_x \models \text{ZFC}$.

Proof. For the even successor levels of the construction we have that properties (i) – (iv) as in the construction are Π_{2n+1}^1 -definable uniformly in any code for the countable premouse \mathcal{M} .

For the odd successor levels recall that U is a Π_{2n+1}^1 -definable set and that F is a Π_{2n+1}^1 -definable function uniformizing U .

Therefore the argument given in the proof of Claim 3 in the proof of Theorem 3.16 yields a contradiction to the background hypothesis that every Σ_{2n+2}^1 -definable sequence of pairwise distinct reals is countable. \square

Moreover we can prove the following claim.

Claim 2. *The model M_x as constructed above has the following properties.*

- (1) $M_x \cap \text{Ord} = \omega_1^V$,
- (2) $x \in M_x$,
- (3) M_x is Σ_{2n+2}^1 -correct in V for real parameters in M_x , that means we have that

$$M_x \prec_{\Sigma_{2n+2}^1} V,$$

- (4) M_x is closed under the operation

$$a \mapsto M_{2n-1}^\#(a),$$

and moreover $M_{2n-1}^\#(a)$ is ω_1 -iterable in M_x for all $a \in M_x$.

Proof. Properties (1) and (2) immediately follow from the construction.

Proof of (3): The proof is organized as an induction on $m < 2n + 1$. We have that M_x is Σ_2^1 -correct in V using Shoenfield's Absoluteness Theorem (see for example Theorem 13.15 in [Ka08]). We assume inductively that M_x is Σ_m^1 -correct in V and prove that M_x is Σ_{m+1}^1 -correct in V . Since the upward implication follows easily as in the proof of Lemma 1.17, we focus on the proof of the downward implication.

So let ψ be a Σ_{m+1}^1 -formula and let $a \in M_x \cap {}^\omega\omega$ be such that $\psi(a)$ holds in V . Say

$$\psi(a) \equiv \exists x \varphi(x, a)$$

for a Π_m^1 -formula φ .

Let $y = \ulcorner \varphi \urcorner \frown a \in {}^\omega\omega$. Then we have that $y \in \text{dom}(F)$ since $\psi(a)$ holds in V and $m+1 \leq 2n+1$. Therefore $\varphi_F(y, F(y))$ holds and $F(y)$ is added to the model M_x at an odd successor level of the construction because we have that $y \in M_x$. Recall that $F(y)$ is chosen such that $(y, F(y)) \in U$, that means we have that

$$F(y) \in U_y = U_{\ulcorner \varphi \urcorner \frown a} = \{x \mid \varphi(x, a)\},$$

by our choice of U . Now the inductive hypothesis implies that

$$M_x \models \varphi(F(y), a)$$

and therefore it follows that $M_x \models \psi(a)$, as desired.

Property (4) now follows by the same argument as the one we gave in the proof of property (3) in Claim 4 in the proof of Theorem 3.16. \square

The following additional property of the model M_x is a key point in proving that $M_{2n}^\#$ exists and is ω_1 -iterable in V .

Claim 3. *For all $x \in {}^\omega\omega$ in the cone of reals given in Theorem 3.27,*

$$M_x \models \text{“}M_{2n}^\# \text{ exists and is } \omega_1\text{-iterable.} \text{”}$$

The proof of this claim is now different from the proof of the analogous claim in the previous section. The reason for this is that at the even levels we cannot assume that we have a $2n$ -suitable premouse to compare the model K^c with (which at the odd levels was given by Lemmas 3.2 and 3.9). This is why we have to give a different argument here.

Proof of Claim 3. Assume this is not the case. Then $(K^c)^{M_x}$ is fully iterable inside M_x by a generalization of Theorem 2.11 in [St96] (here $(K^c)^{M_x}$ denotes the model resulting from a robust-background-extender K^c -construction from [Je03], which is used in [JS13]), since M_x is closed under the operation $a \mapsto M_{2n-1}^\#(a)$. This yields that we can build the core model K^{M_x} inside M_x by a generalization of Theorem 1.1 in [JS13] due to Jensen and Steel. The core model K^{M_x} has to be $2n$ -small, because otherwise we would have that

$$M_x \models \text{“There exists a model which is fully iterable and not } 2n\text{-small”}.$$

This would already imply that $M_{2n}^\#$ exists and is fully iterable inside M_x , so then there is nothing left to show.

Subclaim 1. *K^{M_x} is closed under the operation*

$$a \mapsto M_{2n-1}^\#(a).$$

Proof. We start with considering sets of the form $a = K^{M_x}|\xi$ where $\xi < K^{M_x} \cap \text{Ord}$ is not overlapped by an extender on the K^{M_x} -sequence. That means there is no extender E on the K^{M_x} -sequence such that $\text{crit}(E) \leq \xi < \text{lh}(E)$. We aim to prove that in fact

$$M_{2n-1}^\#(K^{M_x}|\xi) \triangleleft K^{M_x}.$$

We have that the premouse $M_{2n-1}^\#(K^{M_x}|\xi)$ exists inside the model M_x since we have that $\xi < K^{M_x} \cap \text{Ord} = M_x \cap \text{Ord}$ and M_x is closed under the operation

$$a \mapsto M_{2n-1}^\#(a)$$

by property (4) in Claim 2. Consider the coiteration of the premice K^{M_x} and $M_{2n-1}^\#(K^{M_x}|\xi)$ inside M_x . This coiteration takes place above ξ and thus both premice are iterable enough such that the comparison is successful by Lemma 2.11 since $M_{2n-1}^\#(K^{M_x}|\xi)$ is ω_1 -iterable above ξ in M_x . By universality of K^{M_x} inside M_x (by Lemma 3.5 in [St96] applied inside M_x), we have that there is an iterate M^* of $M_{2n-1}^\#(K^{M_x}|\xi)$ and an iterate K^* of K^{M_x} such that $M^* \trianglelefteq K^*$ and the iteration from $M_{2n-1}^\#(K^{M_x}|\xi)$ to M^* is non-dropping on the main branch. Since

$$\rho_\omega(M_{2n-1}^\#(K^{M_x}|\xi)) \leq \xi$$

and since the coiteration takes place above ξ , we have that the iterate M^* of $M_{2n-1}^\#(K^{M_x}|\xi)$ is not sound, if any extender is used on this side of the coiteration. Therefore it follows that in fact

$$M_{2n-1}^\#(K^{M_x}|\xi) \triangleleft K^*.$$

Assume that the K^{M_x} -side moves in the coiteration, that means we have that $K^{M_x} \neq K^*$. Let α be an ordinal such that E_α is the first extender on the K^{M_x} -sequence which is used in the coiteration. Then we have that $\alpha > \xi$. We have in particular that α is a cardinal in K^* . But then since we have that $\rho_\omega(M_{2n-1}^\#(K^{M_x}|\xi)) \leq \xi < \alpha$ and $M_{2n-1}^\#(K^{M_x}|\xi) \triangleleft K^*$, this already implies that

$$\alpha > M_{2n-1}^\#(K^{M_x}|\xi) \cap \text{Ord}.$$

Therefore there was no need to iterate K^{M_x} at all and we have that

$$M_{2n-1}^\#(K^{M_x}|\xi) \triangleleft K^{M_x}.$$

Now let $a \in K^{M_x}$ be arbitrary. Then there exists an ordinal $\xi < K^{M_x} \cap \text{Ord}$ such that $a \in K^{M_x}|\xi$ and ξ is not overlapped by an extender on the K^{M_x} -sequence. We just proved that

$$M_{2n-1}^\#(K^{M_x}|\xi) \triangleleft K^{M_x}.$$

As we argued several times before, by performing a fully backgrounded extender construction (denoted by $L[E](a)_{M_{2n-1}(K^{M_x}|\xi)}$) above a inside the model $M_{2n-1}(K^{M_x}|\xi)$ in the sense of [MS94] (with the smallness hypothesis

weakened) and adding the top extender of the active premouse $M_{2n-1}^\#(K^{M_x|\xi})$ (intersected with $L[E](a)^{M_{2n-1}(K^{M_x|\xi})}$) to an initial segment of the model $L[E](a)^{M_{2n-1}(K^{M_x|\xi})}$ as described in Section 2 in [FNS10] we obtain that

$$M_{2n-1}^\#(a) \in K^{M_x},$$

as desired. \square

Using the Weak Covering Lemma from [MSch95] (see also Theorem 1.1 in [JS13]) we can pick a cardinal $\gamma \in M_x$ such that γ is singular in M_x and γ^+ is computed correctly by K^{M_x} inside M_x . That means we pick γ such that we have

$$(\gamma^+)^{K^{M_x}} = (\gamma^+)^{M_x}.$$

For later purposes we want to pick γ such that it additionally satisfies

$$\text{cf}(\gamma)^{M_x} \geq \omega_1^{M_x}.$$

Subclaim 2. *There exists a real $z \geq_T x$ such that*

- (1) $(\gamma^+)^{M_x} = \omega_2^{Lp^{2n-1}(z)}$, and
- (2) $K^{M_x|(\gamma^+)^{M_x}} \in Lp^{2n-1}(z)$.

Proof. We are going to produce the real z via a five step forcing using an almost disjoint coding. For an introduction into this kind of forcing see for example [FSS14] for a survey or [Sch00] where a similar argument is given.

We force over the ground model

$$Lp^{2n-1}(x, K^{M_x|(\gamma^+)^{M_x}}).$$

We have that $Lp^{2n-1}(x, K^{M_x|(\gamma^+)^{M_x}})$ is a definable subset of M_x because we have by property (4) in Claim 2 that

$$M_{2n-1}^\#(x, K^{M_x|(\gamma^+)^{M_x}}) \in M_x$$

and by Lemma 3.23 the lower part model $Lp^{2n-1}(x, K^{M_x|(\gamma^+)^{M_x}})$ is obtained by iterating the least measure of $M_{2n-1}^\#(x, K^{M_x|(\gamma^+)^{M_x}})$ and its images ω_1^V times and cutting off at ω_1^V .

This implies that in particular

$$\text{cf}(\gamma)^{Lp^{2n-1}(x, K^{M_x|(\gamma^+)^{M_x}})} \geq \omega_1^{M_x}.$$

Step 1: Write $V_0 = Lp^{2n-1}(x, K^{M_x|(\gamma^+)^{M_x}})$ for the ground model. We start with a preparatory forcing that collapses everything below $\omega_1^{M_x}$ to ω . Afterwards we collapse γ to $\omega_1^{M_x}$.

So let $G_0 \in V$ be $\text{Col}(\omega, < \omega_1^{M_x})$ -generic over V_0 and let

$$V'_0 = V_0[G_0].$$

Moreover let $G'_0 \in V$ be $\text{Col}(\omega_1^{M_x}, \gamma)$ -generic over V'_0 and let

$$V_1 = V'_0[G'_0].$$

So we have that $\omega_1^{M_x} = \omega_1^{V_1}$ and by our choice of γ , i.e. $\text{cf}(\gamma)^{V_0} \geq \omega_1^{M_x}$, we moreover have that $(\gamma^+)^{M_x} = (\gamma^+)^{K^{M_x}} = \omega_2^{V_1}$. We write $\omega_1 = \omega_1^{V_1}$ and $\omega_2 = \omega_2^{V_1}$.

Furthermore let A' be a set of ordinals coding G_0 and G'_0 , such that if we let $A \subset (\gamma^+)^{M_x}$ be a code for

$$x \oplus (K^{M_x} | (\gamma^+)^{M_x}) \oplus A',$$

then we have that $G_0, G'_0 \in Lp^{2n-1}(A)$ and $K^{M_x} | (\gamma^+)^{M_x} \in Lp^{2n-1}(A)$.

We can in fact pick the set A such that we have $V_1 = Lp^{2n-1}(A)$ by the following argument: Recall that

$$Lp^{2n-1}(A) = M(A) | \omega_1^V,$$

where $M(A)$ denotes the ω_1^V -th iterate of $M_{2n-1}^\#(A)$ by its least measure and its images for a set A as above. Then we can consider G_0 as being generic over the model $M(x, K^{M_x} | (\gamma^+)^{M_x})$ and G'_0 as being generic over the model $M(x, K^{M_x} | (\gamma^+)^{M_x})[G_0]$, where $M(x, K^{M_x} | (\gamma^+)^{M_x})$ denotes the ω_1^V -th iterate of $M_{2n-1}^\#(x, K^{M_x} | (\gamma^+)^{M_x})$ by its least measure and its images. Since both forcings in Step 1 take place below $(\gamma^+)^{M_x} < \omega_1^V$, it follows as in the proof of Theorem 2.24 that $M(x, K^{M_x} | (\gamma^+)^{M_x})[G_0][G'_0] = M(A)$ for a set $A \subset (\gamma^+)^{M_x}$ coding $x, K^{M_x} | (\gamma^+)^{M_x}, G_0$ and G'_0 and thus we have that $V_1 = M(A) | \omega_1^V$ for this set A , as desired.

Step 2: Before we can perform the first coding using almost disjoint subsets of $\omega_1 = \omega_1^{V_1}$ we have to “reshape” the interval between $(\gamma^+)^{M_x} = \omega_2^{V_1}$ and ω_1 to ensure that the coding we will perform in Step 3 exists. Moreover we have to make sure that the reshaping forcing itself does not collapse ω_1 and $(\gamma^+)^{M_x}$. We are going to show this by proving that the reshaping forcing is $< (\gamma^+)^{M_x}$ -distributive.

We are going to use the following notion of reshaping.

Definition 3.28. *Let η be a cardinal and let $X \subset \eta^+$. We say a function f is (X, η^+) -reshaping iff $f : \alpha \rightarrow 2$ for some $\alpha \leq \eta^+$ and moreover for all $\xi \leq \alpha$ with $\xi < \eta^+$ we have that*

- (i) $L[X \cap \xi, f \upharpoonright \xi] \models |\xi| \leq \eta$, or
- (ii) there is a model N and a Σ_k -elementary embedding

$$j : N \rightarrow Lp^{2n-1}(X) | \eta^{++}$$

for some large enough $k < \omega$ such that

- (a) $\text{crit}(j) = \xi$, $j(\xi) = \eta^+$,
- (b) $\rho_{k+1}(N) \leq \xi$, N is sound above ξ , and
- (c) definably over N there exists a surjection $g : \eta \twoheadrightarrow \xi$.

For future purposes notice that if N is as in Clause (ii) above, then $N \triangleleft Lp^{2n-1}(X \cap \xi)$.

Now we denote with P_1 the forcing that adds an $(A, (\gamma^+)^{M_x})$ -reshaping function for $(\gamma^+)^{M_x} = \omega_2^{V_1}$, defined inside our new ground model $V_1 = Lp^{2n-1}(A)$.

We let $p \in P_1$ iff p is an $(A, (\gamma^+)^{M_x})$ -reshaping function with $\text{dom}(p) < (\gamma^+)^{M_x}$ and we order two conditions p and q in P_1 by reverse inclusion, that means we let $p \leq_{P_1} q$ iff $q \subseteq p$.

First notice that the forcing P_1 is extendible, that means for every ordinal $\alpha < (\gamma^+)^{M_x}$ the set $D^\alpha = \{p \in P_1 \mid \text{dom}(p) \geq \alpha\}$ is open and dense in P_1 . In fact, for each $p \in P_1$ and every $\alpha < (\gamma^+)^{M_x}$ there is some $q \leq_{P_1} p$ such that $\text{dom}(q) \geq \alpha$ and $L[X \cap \xi, p \upharpoonright \xi] \models |\xi| \leq \eta$ for all ξ with $\text{dom}(p) < \xi \leq \text{dom}(q)$. We now want to show that P_1 is $< (\gamma^+)^{M_x}$ -distributive. For that we fix a condition $p \in P_1$ and open dense sets $(D_\beta \mid \beta < \omega_1)$. We aim to find a condition $q \leq_{P_1} p$ such that $q \in D_\beta$ for all $\beta < \omega_1$.

Consider, for some large enough fixed natural number k , transitive Σ_k -elementary substructures of the model $Lp^{2n-1}(A) = V_1$. More precisely we want to pick a continuous sequence

$$(N_\alpha, \pi_\alpha, \xi_\alpha \mid \alpha \leq \omega_1)$$

of transitive models N_α of size $|\omega_1^{V_1}|$ together with Σ_k -elementary embeddings

$$\pi_\alpha : N_\alpha \rightarrow Lp^{2n-1}(A)$$

and an increasing sequence of ordinals ξ_α such that we have $p \in N_0$, and for all $\alpha \leq \omega_1$

- (1) $\text{crit}(\pi_\alpha) = \xi_\alpha$ with $\pi_\alpha(\xi_\alpha) = (\gamma^+)^{M_x}$,
- (2) for all ordinals $\alpha < \omega_1$ we have that $\rho_{k+1}(N_\alpha) \leq \xi_\alpha$ and N_α is sound above ξ_α , and
- (3) $\{p\} \cup \{D_\beta \mid \beta < \omega_1\} \subset \text{ran}(\pi_\alpha)$.

We can obtain N_α and π_α for all $\alpha \leq \omega_1$ with these properties inductively as follows. Let M_0 be the (uncollapsed) Σ_k -hull of

$$\gamma \cup \{p\} \cup \{D_\beta \mid \beta < \omega_1\}$$

taken inside $Lp^{2n-1}(A)$. Then let N_0 be the Mostowski collapse of M_0 and let

$$\pi_0 : N_0 \rightarrow M_0 \prec_{\Sigma_k} Lp^{2n-1}(A)$$

be the inverse of the embedding obtained from the Mostowski collapse with critical point ξ_0 .

Now assume we already constructed $(N_\alpha, \pi_\alpha, \xi_\alpha)$ and M_α for some $\alpha < \omega_1$. Then we let $M_{\alpha+1}$ be the (uncollapsed) Σ_k -hull of

$$\gamma \cup \{p\} \cup \{D_\beta \mid \beta < \omega_1\} \cup M_\alpha \cup \{M_\alpha\}$$

taken inside $Lp^{2n-1}(A)$. Further let $N_{\alpha+1}$ be the Mostowski collapse of $M_{\alpha+1}$ and let

$$\pi_{\alpha+1} : N_{\alpha+1} \rightarrow M_{\alpha+1} \prec_{\Sigma_k} Lp^{2n-1}(A)$$

be the inverse of the embedding obtained from the Mostowski collapse with critical point $\xi_{\alpha+1}$. Note that we have $\xi_{\alpha+1} > \xi_\alpha$.

Moreover if we assume that $(N_\alpha, \pi_\alpha, \xi_\alpha)$ is already constructed for all $\alpha < \lambda$ for some limit ordinal $\lambda \leq \omega_1$, then we let

$$N_\lambda = \bigcup_{\alpha < \lambda} N_\alpha,$$

$$\pi_\lambda = \bigcup_{\alpha < \lambda} \pi_\alpha,$$

and $\xi_\lambda = \text{crit}(\pi_\lambda)$.

Recall that we fixed open dense sets $(D_\beta \mid \beta < \omega_1)$. We are now going to construct a sequence $(p_\alpha \mid \alpha \leq \omega_1)$ of conditions such that $p_{\alpha+1} \leq_{P_1} p_\alpha$ and $p_{\alpha+1} \in D_\alpha$ for all $\alpha < \omega_1$. Moreover we are going to construct these conditions such that we inductively maintain $p_\alpha \in \pi_\alpha^{-1}(P_1) \subset N_\alpha$.

We start with $p_0 = p \in N_0$. For the successor step suppose that we already defined $p_\alpha \in \pi_\alpha^{-1}(P_1) \subset N_\alpha$ for some $\alpha < \omega_1$. Then we have that $\text{dom}(p_\alpha) < \xi_\alpha$ and $p_\alpha \in N_{\alpha+1}$ by the definition of the models and embeddings $(N_\alpha, \pi_\alpha \mid \alpha \leq \omega_1)$. By extendibility of the forcing $\pi_{\alpha+1}^{-1}(P_1)$ and the density of the set $\pi_{\alpha+1}^{-1}(D_\alpha) \subseteq D_\alpha$, there exists a condition $p_{\alpha+1} \leq_{P_1} p_\alpha$ such that we have $p_{\alpha+1} \in \pi_{\alpha+1}^{-1}(P_1) \subset N_{\alpha+1}$, $p_{\alpha+1} \in D_\alpha$ and $\text{dom}(p_{\alpha+1}) \geq \xi_\alpha$ since $\pi_{\alpha+1}^{-1}(P_1) \subseteq P_1$.

For a limit ordinal $\lambda \leq \omega_1$ we simply let $p_\lambda = \bigcup_{\alpha < \lambda} p_\alpha$. We have that the sequence $(\xi_\alpha \mid \alpha < \lambda)$ of critical points of $(\pi_\alpha \mid \alpha < \lambda)$ is definable over N_λ since for $\alpha < \lambda$ the model N_α is equal to the transitive collapse of a Σ_k -elementary submodel of N_λ which is constructed inside N_λ exactly as it was constructed inside $Lp^{2n-1}(A)$ above. Therefore we have that

$$\text{cf}^{N_\lambda}(\xi_\lambda) \leq \lambda \leq \omega_1 = \omega_1^{V_1}$$

which implies that

$$N_\lambda \models |\xi_\lambda| \leq \omega_1^{V_1}.$$

As $\text{dom}(p_\lambda) = \xi_\lambda$, this buys us that p_λ is in fact a condition in the forcing P_1 .

Now consider the function $q = p_{\omega_1}$. Then $q \in P_1$ and $q \in D_\beta$ for all $\beta < \omega_1$. We have shown that the reshaping forcing P_1 is $< (\gamma^+)^{M_x}$ -distributive and therefore does not collapse ω_1 and $(\gamma^+)^{M_x} = \omega_2$.

So let G_1 be P_1 -generic over V_1 and let $V_2 = V_1[G_1]$. The extendibility of the forcing P_1 yields that $\bigcup G_1$ is an $(A, (\gamma^+)^{M_x})$ -reshaping function with domain $(\gamma^+)^{M_x}$. Let B' be a subset of $(\gamma^+)^{M_x}$ which codes the function $\bigcup G_1$, for example the subset of $(\gamma^+)^{M_x}$ which has $\bigcup G_1$ as its characteristic function. Finally let $B \subset (\gamma^+)^{M_x}$ be a code for $A \oplus B'$.

As at the end of Step 1 we can pick this code $B \subset (\gamma^+)^{M_x}$ such that the model V_2 is of the form $Lp^{2n-1}(B)$ by the following argument: Recall that

$$Lp^{2n-1}(B) = M(B)|\omega_1^V,$$

where as above $M(B)$ denotes the ω_1^V -th iterate of $M_{2n-1}^\#(B)$ by its least measure and its images. Therefore we can consider G_1 as being generic over $M(A)$. This yields analogous to the argument at the end of Step 1 that we can pick B such that $V_2 = M(B)|\omega_1^V$ because the “reshaping forcing” P_1 takes place below $(\gamma^+)^{M_x} < \omega_1^V$. Therefore we have that

$$V_2 = Lp^{2n-1}(B).$$

Step 3: Now we can perform the first coding using almost disjoint subsets of $\omega_1 = \omega_1^{V_2} = \omega_1^{V_1}$. Since B is “reshaped” we can inductively construct a sequence of almost disjoint subsets of ω_1 ,

$$(A_\xi \mid \xi < (\gamma^+)^{M_x}),$$

as follows. Let $\xi < (\gamma^+)^{M_x}$ be such that we already constructed a sequence $(A_\zeta \mid \zeta < \xi)$ of almost disjoint subsets of ω_1 .

Case 1. $L[B \cap \xi] \models |\xi| \leq \omega_1^{V_2}$.

Then we let A_ξ be the least subset of ω_1 in $L[B \cap \xi]$ which is almost disjoint from any A_ζ for $\zeta < \xi$ and which satisfies that

$$|\omega_1 \setminus \bigcup_{\zeta \leq \xi} A_\zeta| = \aleph_1.$$

Case 2. Otherwise.

Let N be the least initial segment of $Lp^{2n-1}(A \cap \xi)^{Lp^{2n-1}(B)}$ such that $\rho_\omega(N) \leq \xi$, N is sound above ξ , ξ is the largest cardinal in N , and definably over N there exists a surjection $g : \omega_1^{V_2} \rightarrow \xi$. Now let A_ξ be the least subset of $\omega_1^{V_2}$ which is definable over N , almost disjoint from any A_ζ for $\zeta < \xi$ and which satisfies that $|\omega_1 \setminus \bigcup_{\zeta \leq \xi} A_\zeta| = \aleph_1$.

That the set A_ξ be well-defined in this case follows from the fact that the set $B \subset (\gamma^+)^{M_x}$ is “reshaped” by the following argument. As B is “reshaped” we have in Case 2 above that there exists a model N as in Definition 3.28 (ii). We have that $N \triangleleft Lp^{2n-1}(A \cap \xi)$. In general it need not be the case that $Lp^{2n-1}(A \cap \xi)^{Lp^{2n-1}(B)}$ is equal to $Lp^{2n-1}(A \cap \xi)$ (see Lemma 3.25), but as ξ is the largest cardinal in N , it follows that in fact $N \triangleleft Lp^{2n-1}(A \cap \xi)^{Lp^{2n-1}(B)}$. Hence any N witnessing at ξ that B is “reshaped” is an initial segment of $Lp^{2n-1}(A \cap \xi)^{Lp^{2n-1}(B)}$, so that the set A_ξ is indeed well-defined.

The sequence $(A_\xi \mid \xi < (\gamma^+)^{M_x})$ is now definable in $V_2 = Lp^{2n-1}(B)$.

Now let P_2 be the forcing for coding B by a subset of ω_1 using the almost disjoint sets $(A_\xi \mid \xi < (\gamma^+)^{M_x})$. That means a condition $p \in P_2$ is a pair

(p_l, p_r) such that $p_l : \alpha \rightarrow 2$ for some $\alpha < \omega_1$ and p_r is a countable subset of $(\gamma^+)^{M_x}$. We say $p = (p_l, p_r) \leq_{P_2} (q_l, q_r) = q$ iff $q_l \subseteq p_l$, $q_r \subseteq p_r$, and for all $\xi \in q_r$ we have that if $\xi \in B$, then

$$\{\beta \in \text{dom}(p_l) \setminus \text{dom}(q_l) \mid p_l(\beta) = 1\} \cap A_\xi = \emptyset.$$

An easy argument shows that the $(\gamma^+)^{M_x}$ -c.c. holds true for the forcing P_2 . Moreover it is ω -closed and therefore no cardinals are collapsed.

Let G_2 be P_2 -generic over V_2 and let

$$C' = \bigcup_{p \in G_2} \{\beta \in \text{dom}(p_l) \mid p_l(\beta) = 1\}.$$

Then $C' \subset \omega_1$ and we have that for all $\xi < (\gamma^+)^{M_x}$,

$$\xi \in B \text{ iff } |C' \cap A_\xi| \leq \aleph_0.$$

Finally let $V_3 = V_2[G_2]$. By the same argument as we gave at the end of Step 2 we can obtain that

$$V_3 = Lp^{2n-1}(C)$$

for some set $C \subset \omega_1$ coding C' and the real x , as the model $Lp^{2n-1}(C)$ can successfully decode the set $B \subset (\gamma^+)^{M_x}$ by the following argument. We show inductively that for every $\xi < (\gamma^+)^{M_x}$, $(A_\zeta \mid \zeta < \xi) \in Lp^{2n-1}(C)$ and $B \cap \xi \in Lp^{2n-1}(C)$. This yields that $B \in Lp^{2n-1}(C)$.

For the inductive step let $\xi < (\gamma^+)^{M_x}$ be an ordinal and assume inductively that we have

$$(A_\zeta \mid \zeta < \xi) \in Lp^{2n-1}(C).$$

Since for all $\zeta < \xi$,

$$\zeta \in B \text{ iff } |C' \cap A_\zeta| \leq \aleph_0,$$

we have that $B \cap \xi \in Lp^{2n-1}(C)$.

In Case 1, i.e. if $L[B \cap \xi] \models |\xi| \leq \omega_1^{V_2}$, the set A_ξ can easily be identified inside $Lp^{2n-1}(C)$. In Case 2 let N be the least initial segment of $Lp^{2n-1}(A \cap \xi)^{Lp^{2n-1}(C)}$ such that $\rho_\omega(N) \leq \xi$, N is sound above ξ , ξ is the largest cardinal in N , and definably over N there exists a surjection $g : \omega_1 \twoheadrightarrow \xi$. Such an N exists due to the fact that B is “reshaped”: there is even some $N \triangleleft Lp^{2n-1}(A \cap \xi)$ such that $\rho_\omega(N) \leq \xi$, N is sound above ξ , ξ is the largest cardinal in N , and definably over N there exists a surjection $g : \omega_1 \twoheadrightarrow \xi$; and $Lp^{2n-1}(A \cap \xi)^{Lp^{2n-1}(C)} \subseteq Lp^{2n-1}(A \cap \xi)$. Thus the least initial segment N of $Lp^{2n-1}(A \cap \xi)^{Lp^{2n-1}(C)}$ with these properties exist and it will also be an initial segment of $Lp^{2n-1}(A \cap \xi)$, and N will then also be an initial segment of $Lp^{2n-1}(A \cap \xi)^{Lp^{2n-1}(B)}$ (for the B as above) which was used to identify A_ξ . We have shown that in each case A_ξ can be identified inside $Lp^{2n-1}(C)$.

As the identification of the next A_ξ was according to a uniform procedure, we get that in fact

$$(A_\zeta \mid \zeta \leq \xi) \in Lp^{2n-1}(C).$$

Step 4: Before we can “code down to a real”, that means before we can find a real z such that $K^{M_x} | (\gamma^+)^{M_x} \in Lp^{2^{n-1}}(z)$, we have to perform another “reshaping” similar to the one in Step 2. So let P_3 be the forcing for adding a (C, ω_1) -reshaping function working in V_3 as the new ground model, where $\omega_1 = \omega_1^{V_3} = \omega_1^{V_2}$. That means we let $p \in P_3$ iff p is a (C, ω_1) -reshaping function with $\text{dom}(p) < \omega_1$. The order of two conditions p and q in P_3 is again by reverse inclusion, that means $p \leq_{P_3} q$ iff $q \subseteq p$.

The forcing P_3 is extendable and $< \omega_1$ -distributive by the same arguments as we gave in Step 2 since we have that $V_3 = Lp^{2^{n-1}}(C)$. Therefore P_3 does not collapse ω_1 .

Let G_3 be P_3 -generic over V_3 and let $V_4 = V_3[G_3]$. We again have that $\bigcup G_3$ is a (C, ω_1) -reshaping function with domain ω_1 , because P_3 is extendable. Let D' be a subset of ω_1 which codes $\bigcup G_3$, for example the subset of ω_1 which has $\bigcup G_3$ as its characteristic function. Finally let $D \subset \omega_1$ code $C \oplus D'$.

By the same argument as the one we gave at the end of Step 2 we can obtain that in fact

$$V_4 = Lp^{2^{n-1}}(D).$$

Step 5: Now we are ready to finally “code down to a real”. Since D is “reshaped” we can consider a uniformly defined sequence

$$(B_\xi \mid \xi < \omega_1)$$

of almost disjoint subsets of ω analogous to Step 3, where $\omega_1 = \omega_1^{V_4} = \omega_1^{V_3}$. Now we let P_4 be the forcing for coding D by a subset of ω using the almost disjoint sets $(B_\xi \mid \xi < \omega_1)$. That means a condition $p \in P_4$ is a pair (p_l, p_r) such that $p_l : \alpha \rightarrow 2$ for some $\alpha < \omega$ and p_r is a finite subset of ω_1 . We say $p = (p_l, p_r) \leq_{P_4} (q_l, q_r) = q$ iff $q_l \subseteq p_l$, $q_r \subseteq p_r$, and for all $\xi \in q_r$ we have that if $\xi \in D$, then

$$\{\beta \in \text{dom}(p_l) \setminus \text{dom}(q_l) \mid p_l(\beta) = 1\} \cap B_\xi = \emptyset.$$

As in Step 3 above an easy argument shows that the forcing P_4 has the c.c.c. and therefore no cardinals are collapsed.

Finally let G_4 be P_4 -generic over V_4 and let

$$E' = \bigcup_{p \in G_4} \{\beta \in \text{dom}(p_l) \mid p_l(\beta) = 1\}.$$

Then $E' \subset \omega$ and we have that for all $\xi < \omega_1$,

$$\xi \in D \text{ iff } |E' \cap B_\xi| < \aleph_0.$$

Let $V_5 = V_4[G_4]$ and finally let z be a real coding E' and the real x . Analogous to the arguments given at the end of Step 3 we can pick the real $z \geq_T x$ such that we have

$$V_5 = Lp^{2^{n-1}}(z)$$

and the model $Lp^{2n-1}(z)$ is able to successfully decode the set D and thereby also the set A .

This ultimately yields that we have a real $z \geq_T x$ such that

$$(\gamma^+)^{M_x} = \omega_2^{Lp^{2n-1}(z)}$$

and

$$K^{M_x} | (\gamma^+)^{M_x} \in Lp^{2n-1}(z).$$

□

Subclaim 3. $K^{M_x} | (\gamma^+)^{M_x}$ is fully iterable inside $Lp^{2n-1}(z)$.

Proof. It is enough to show that the $2n$ -small premouse $K^{M_x} | (\gamma^+)^{M_x}$ is ω_1 -iterable inside $Lp^{2n-1}(z)$ because once we showed this, an absoluteness argument, as for example similar to the one we already gave in the proof of Lemma 3.9, yields that $K^{M_x} | (\gamma^+)^{M_x}$ is in fact fully iterable inside $Lp^{2n-1}(z)$ since the iteration strategy for $K^{M_x} | (\gamma^+)^{M_x}$ is given by \mathcal{Q} -structures $\mathcal{Q}(\mathcal{T})$ for iteration trees \mathcal{T} on $K^{M_x} | (\gamma^+)^{M_x}$ which are $(2n-1)$ -small above $\delta(\mathcal{T})$ and such \mathcal{Q} -structures $\mathcal{Q}(\mathcal{T})$ are contained in every lower part model at the level $2n-1$, as for example $Lp^{2n-1}(z)$, by definition of the lower part model.

In a first step we show that there is a tree T such that $p[T]$ is a universal Π_{2n}^1 -set in $Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})$ and moreover for every forcing \mathbb{P} of size at most $(\gamma^+)^{M_x}$ in $Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})$ and every \mathbb{P} -generic G over $Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})$ we have that

$$Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})[G] \models \text{“}p[T] \text{ is a universal } \Pi_{2n}^1\text{-set”}.$$

Let φ be a Π_{2n}^1 -formula defining a universal Π_{2n}^1 -set, i.e. $\{y \in {}^\omega\omega \mid \varphi(y)\}$ is a universal Π_{2n}^1 -set. Then we let $T \in Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})$ be a tree of height ω searching for $y, H, \mathcal{M}, \sigma, \mathbb{Q}$ and g such that

- (1) $y \in {}^\omega\omega$,
- (2) \mathcal{M} is a countable (x, H) -premouse,
- (3) $\sigma : \mathcal{M} \rightarrow Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})$ is a sufficiently elementary embedding,
- (4) $\sigma(H) = K^{M_x} | (\gamma^+)^{M_x}$, and
- (5) $\mathbb{Q} \in \mathcal{M}$ is a partial order of size at most $H \cap \text{Ord}$ in \mathcal{M} and g is \mathbb{Q} -generic over \mathcal{M} such that

$$\mathcal{M}[g] \models \varphi(y).$$

This tree T has the properties we claimed above by the following argument. The model $Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})$ is closed under the operation $a \mapsto M_{2n-2}^\#(a)$ and is therefore Σ_{2n}^1 -correct in V . The same holds for the model $Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})[G]$, where G is \mathbb{P} -generic over the model $Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})$ for some forcing \mathbb{P} of size at most $(\gamma^+)^{M_x}$ in the model $Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})$. Moreover we have that if $\mathcal{M}, H, \sigma, \mathbb{Q}$ and g are as searched by the tree T , then the forcing $\mathbb{Q} \in \mathcal{M}$ has size at most $H \cap \text{Ord}$ in \mathcal{M} and the embedding $\sigma : \mathcal{M} \rightarrow Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})$ is

sufficiently elementary, so it follows that $\mathcal{M}[g]$ is Σ_{2n}^1 -correct in V . This easily yields that $p[T]$ is a universal Π_{2n}^1 -set in $Lp^{2n-1}(x, K^{M_x|(\gamma^+)^{M_x}})[G]$ for every generic set G as above, in fact

$$p[T] \cap Lp^{2n-1}(x, K^{M_x|(\gamma^+)^{M_x}})[G] = \{y \in {}^\omega\omega \mid \varphi(y)\} \cap Lp^{2n-1}(x, K^{M_x|(\gamma^+)^{M_x}})[G].$$

In a second step we now define another tree U whose well-foundedness is going to witness that $K^{M_x|(\gamma^+)^{M_x}}$ is ω_1 -iterable. So we define the tree $U \in Lp^{2n-1}(x, K^{M_x|(\gamma^+)^{M_x}})$ such that U is searching for $\bar{K}, \pi, \mathcal{T}, \mathcal{N}, \sigma$ and a sequence $(\mathcal{Q}_\lambda \mid \lambda \leq \text{lh}(\mathcal{T}), \lambda \text{ limit})$ with the following properties.

- (1) \bar{K} is a countable premouse,
- (2) $\pi : \bar{K} \rightarrow K^{M_x|(\gamma^+)^{M_x}}$ is an elementary embedding,
- (3) \mathcal{T} is a countable putative⁵ iteration tree on \bar{K} such that for all limit ordinals $\lambda < \text{lh}(\mathcal{T})$,

$$\mathcal{Q}_\lambda \trianglelefteq \mathcal{M}_\lambda^{\mathcal{T}},$$

- (4) for all limit ordinals $\lambda \leq \text{lh}(\mathcal{T})$, \mathcal{Q}_λ is a Π_{2n}^1 -iterable (above $\delta(\mathcal{T} \upharpoonright \lambda)$) \mathcal{Q} -structure for $\mathcal{T} \upharpoonright \lambda$ and \mathcal{Q}_λ is $(2n-1)$ -small above $\delta(\mathcal{T} \upharpoonright \lambda)$ (where this Π_{2n}^1 -statement is witnessed using the tree T defined above),
- (5) \mathcal{N} is a countable model of ZFC^- such that either

$$\mathcal{N} \models \text{“}\mathcal{T} \text{ has a last ill-founded model”},$$

or else

$$\mathcal{N} \models \text{“}\text{lh}(\mathcal{T}) \text{ is a limit ordinal and there is no cofinal branch } b \text{ through } \mathcal{T} \text{ such that } \mathcal{Q}_{\text{lh}(\mathcal{T})} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}\text{”},$$

and

- (6) $\sigma : \mathcal{N} \cap \text{Ord} \hookrightarrow \omega_1$.

Here a code for the sequence $(\mathcal{Q}_\lambda \mid \lambda \leq \text{lh}(\mathcal{T}), \lambda \text{ limit})$ of \mathcal{Q} -structures satisfying property (4) can be read off from $p[T]$.

Recall that we have that the core model K^{M_x} is $2n$ -small and thereby fully iterable in the model M_x via an iteration strategy which is guided by \mathcal{Q} -structures $\mathcal{Q}(\mathcal{T})$ for iteration trees \mathcal{T} on K^{M_x} of limit length which are $(2n-1)$ -small above $\delta(\mathcal{T})$. This implies that the premouse $K^{M_x|(\gamma^+)^{M_x}}$ is iterable inside the model $Lp^{2n-1}(x, K^{M_x|(\gamma^+)^{M_x}})$ (which as argued earlier is a definable subset of M_x) by an argument as in the proof of Lemma 2.11, because $K^{M_x|(\gamma^+)^{M_x}}$ is a $2n$ -small premouse and the ω_1 -iterable \mathcal{Q} -structures $\mathcal{Q}(\mathcal{T})$ for iteration trees \mathcal{T} on $K^{M_x|(\gamma^+)^{M_x}}$ in $Lp^{2n-1}(x, K^{M_x|(\gamma^+)^{M_x}})$ are contained in the model $Lp^{2n-1}(x, K^{M_x|(\gamma^+)^{M_x}})$ as they are $(2n-1)$ -small above $\delta(\mathcal{T})$.

⁵Recall that we say that a tree \mathcal{T} is a *putative iteration tree* if \mathcal{T} satisfies all properties of an iteration tree, but we allow the last model of \mathcal{T} to be ill-founded, in case \mathcal{T} has a last model.

We aim to show that the tree U defined above is well-founded inside the model $Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})$. So assume toward a contradiction that U is ill-founded in $Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})$ and let $\bar{K}, \pi, \mathcal{T}, \mathcal{N}, \sigma$ and $(\mathcal{Q}_\lambda \mid \lambda \leq \text{lh}(\mathcal{T}), \lambda \text{ limit})$ be as above satisfying properties (1) – (6) in the definition of the tree U .

Assume that the iteration tree \mathcal{T} has limit length since the other case is easier. Since as argued above the premouse $K^{M_x} | (\gamma^+)^{M_x}$ is countably iterable inside $Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})$, we have that in $Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})$ there exists a cofinal well-founded branch b through the iteration tree \mathcal{T} on \bar{K} and ω_1 -iterable \mathcal{Q} -structures $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ for all limit ordinals $\lambda \leq \text{lh}(\mathcal{T})$ such that we have $\mathcal{Q}(\mathcal{T}) = \mathcal{Q}(b, \mathcal{T})$ by an argument analogous to the one we gave in Case 1 in the proof of Lemma 3.9. Moreover every \mathcal{Q} -structure $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ is $(2n - 1)$ -small above $\delta(\mathcal{T} \upharpoonright \lambda)$.

Since the model $Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})$ is closed under the operation $a \mapsto M_{2n-2}^\#(a)$, the proofs of Lemmas 2.11 and 2.12 imply that we can successfully compare $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ and \mathcal{Q}_λ for all limit ordinals $\lambda \leq \text{lh}(\mathcal{T})$. This implies that in fact

$$\mathcal{Q}(\mathcal{T} \upharpoonright \lambda) = \mathcal{Q}_\lambda.$$

Therefore it follows by an absoluteness argument as the one we already gave in the proof of Lemma 2.11 that

$\mathcal{N} \models$ “there exists a cofinal branch b through \mathcal{T} such that

$$\mathcal{Q}_{\text{lh}(\mathcal{T})} \trianglelefteq \mathcal{M}_b^{\mathcal{T}},$$

because we in particular have that $\mathcal{Q}_{\text{lh}(\mathcal{T})} = \mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$.

Therefore the tree U is well-founded in the model $Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})$ and by absoluteness of well-foundedness this implies that the tree U is also well-founded inside $Lp^{2n-1}(z)$.

We have by construction of the tree T that a code for the sequence $(\mathcal{Q}_\lambda \mid \lambda \leq \text{lh}(\mathcal{T}), \lambda \text{ limit})$ can still be read off from $p[T]$ in $Lp^{2n-1}(z)$ because the forcing we performed over the ground model $Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})$ to obtain the real z has size at most $(\gamma^+)^{M_x}$ in $Lp^{2n-1}(x, K^{M_x} | (\gamma^+)^{M_x})$. Therefore the well-foundedness of the tree U in $Lp^{2n-1}(z)$ implies that $K^{M_x} | (\gamma^+)^{M_x}$ is ω_1 -iterable in $Lp^{2n-1}(z)$. \square

Following [MSch04], we now aim to make sense of

$$(K^c)^{Lp^{2n-1}(z)}.$$

The issue is that the definition of *certified* in the construction of the model K^c in the sense of [MSch04] makes reference to some fixed class of ordinals A such that $V = L[A]$. Therefore the model K^c constructed in this sense is in general not contained in HOD, because whether an extender is certified in the sense of Definition 1.6 in [MSch04] may depend on the choice of A . The following claim is supposed to express the fact that we may choose A with

$Lp^{2n-1}(z) \models \text{“}V = L[A]\text{”}$ in such a way that the resulting K^c is a subclass of HOD from the point of view of $Lp^{2n-1}(z)$.

Subclaim 4. $(K^c)^{Lp^{2n-1}(z)} \subset \text{HOD}^{Lp^{2n-1}(z)}$.

Proof. Let $z' \in Lp^{2n-1}(z)$ be an arbitrary real with $z \leq_T z'$. In particular, $Lp^{2n-1}(z) = Lp^{2n-1}(z')$ by Lemma 3.24. Let us write $A_{z'} = z' \frown E_{z'}$, where $E_{z'}$ codes the extender sequence of $Lp^{2n-1}(z')$. We may construe $A_{z'}$ as a set of ordinals in a canonical fashion.

Claim 1. *Let $z' \in Lp^{2n-1}(z)$ be a real with $z \leq_T z'$. An extender E is certified with respect to A_z iff E is certified with respect to $A_{z'}$, in the sense of Definition 1.6 in [MSch04].*

Proof. It follows from Lemma 3.24 that

$$Lp^{2n-1}(z') = (Lp^{2n-1}(z'))^{Lp^{2n-1}(z)}$$

and we even have that the set $A_{z'} \cap \omega_1^{Lp^{2n-1}(z)}$ may be easily computed from $A_z \cap \omega_1^{Lp^{2n-1}(z)}$. Symmetrically, we have that the set $A_z \cap \omega_1^{Lp^{2n-1}(z)}$ may be easily computed from $A_{z'} \cap \omega_1^{Lp^{2n-1}(z)}$. Moreover the extender sequences of $Lp^{2n-1}(z)$ and $Lp^{2n-1}(z')$ literally agree above $\omega_1^{Lp^{2n-1}(z)}$. Therefore it follows easily that an extender E is certified with respect to A_z iff E is certified with respect to $A_{z'}$. \square

We have shown that inside $Lp^{2n-1}(z)$ there is a cone of reals z' such that the output of the K^c construction performed à la [MSch04] and using $A_{z'}$ is not sensitive to the particular choice of z' from that cone. Hence if we let

$$(K^c)^{Lp^{2n-1}(z)}$$

be this common output, then

$$(K^c)^{Lp^{2n-1}(z)} \subset \text{HOD}^{Lp^{2n-1}(z)},$$

as desired. \square

In what follows we will also need the following notion of iterability.

Definition 3.29. *Let N be a countable premouse. Then we inductively define an iteration strategy Λ for N as follows. Assume that \mathcal{T} is a normal iteration tree on N of limit length according to Λ such that there exists a \mathcal{Q} -structure $\mathcal{Q} \supseteq \mathcal{M}(\mathcal{T})$ for \mathcal{T} which is fully iterable above $\delta(\mathcal{T})$. Then we define that $\Lambda(\mathcal{T}) = b$ iff b is a cofinal branch through \mathcal{T} such that either*

- (i) $\mathcal{Q} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$, or
- (ii) b does not drop (so in particular the iteration embedding $i_b^{\mathcal{T}}$ exists), there exists an ordinal $\delta < N \cap \text{Ord}$ such that $i_b^{\mathcal{T}}(\delta) = \delta(\mathcal{T})$ and

$$N \models \text{“}\delta \text{ is Woodin”},$$

and there exists a $\tilde{Q} \supseteq N|\delta$ such that

$$\tilde{Q} \models \text{“}\delta \text{ is Woodin”},$$

but δ is not definably Woodin over \tilde{Q} and if we lift the iteration tree \mathcal{T} on \tilde{Q} , call this iteration tree \mathcal{T}^* , then

$$i_b^{\mathcal{T}^*} : \tilde{Q} \rightarrow \mathcal{Q}.$$

Definition 3.30. Let N be a countable premouse. Then we say that N is \mathcal{Q} -structure iterable iff for every iteration tree \mathcal{T} on N which is according to the iteration strategy Λ from Definition 3.29 the following holds.

- (i) If \mathcal{T} has limit length and there exists a \mathcal{Q} -structure $\mathcal{Q} \supseteq \mathcal{M}(\mathcal{T})$ for \mathcal{T} which is fully iterable above $\delta(\mathcal{T})$, then there exists a cofinal well-founded branch b through \mathcal{T} such that $\Lambda(\mathcal{T}) = b$.
- (ii) If \mathcal{T} has a last model, then every putative iteration tree \mathcal{U} extending \mathcal{T} such that $\text{lh}(\mathcal{U}) = \text{lh}(\mathcal{T}) + 1$ has a well-founded last model.

The premouse

$$(K^c)^{Lp^{2n-1}(z)}$$

is countably iterable in $Lp^{2n-1}(z)$ by the iterability proof of Chapter 9 in [St96] adapted as in Section 2 in [MSch04].

Subclaim 5. In $Lp^{2n-1}(z)$,

$$(K^c)^{Lp^{2n-1}(z)} \text{ is } \mathcal{Q}\text{-structure iterable.}$$

Proof. Assume there exists an iteration tree \mathcal{T} on $(K^c)^{Lp^{2n-1}(z)}$ which witnesses that $(K^c)^{Lp^{2n-1}(z)}$ is not \mathcal{Q} -structure iterable inside $Lp^{2n-1}(z)$. Since the other case is easier assume that \mathcal{T} has limit length. That means in particular that there exists a \mathcal{Q} -structure $\mathcal{Q}(\mathcal{T}) \supseteq \mathcal{M}(\mathcal{T})$ for \mathcal{T} which is fully iterable above $\delta(\mathcal{T})$, but there is no cofinal well-founded branch b through \mathcal{T} in $Lp^{2n-1}(z)$ such that $\Lambda(\mathcal{T}) = b$.

For some large enough natural number m let H be the Mostowski collapse of $Hull_m^{Lp^{2n-1}(z)}$ such that H is sound and $\rho_\omega(H) = \rho_{m+1}(H) = \omega$. Furthermore let

$$\pi : H \rightarrow Lp^{2n-1}(z)$$

be the uncollapse map such that $\mathcal{T}, \mathcal{Q}(\mathcal{T}) \in \text{ran}(\pi)$. Moreover let $\bar{\mathcal{T}}, \bar{\mathcal{Q}} \in H$ be such that $\pi(\bar{\mathcal{T}}) = \mathcal{T}$, $\pi(\bar{\mathcal{Q}}) = \mathcal{Q}(\mathcal{T})$, and let $\bar{K} \subset H$ be such that $\pi(\bar{K}|\gamma) = (K^c)^{Lp^{2n-1}(z)}|\pi(\gamma)$ for any $\gamma < H \cap \text{Ord}$. That means in particular that $\bar{\mathcal{T}}$ is an iteration tree on \bar{K} .

As argued above we have that $(K^c)^{Lp^{2n-1}(z)}$ is countably iterable in $Lp^{2n-1}(z)$. Therefore there exists a cofinal well-founded branch \bar{b} through $\bar{\mathcal{T}}$ in $Lp^{2n-1}(z)$.

Now we consider two different cases.

Case 1. There is a drop along the branch \bar{b} .

In this case there exists a \mathcal{Q} -structure $\mathcal{Q}^* \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$ for $\bar{\mathcal{T}}$, because there is a drop along \bar{b} . A standard comparison argument shows that $\mathcal{Q}^* = \bar{\mathcal{Q}}$ and thus $\bar{\mathcal{Q}} \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$ is a \mathcal{Q} -structure for $\bar{\mathcal{T}}$.

Now consider the statement

$$\phi(\bar{\mathcal{T}}, \bar{\mathcal{Q}}) \equiv \text{“there is a cofinal branch } b \text{ through } \bar{\mathcal{T}} \text{ such that } \bar{\mathcal{Q}} \trianglelefteq \mathcal{M}_b^{\bar{\mathcal{T}}} \text{”}.$$

This statement $\phi(\bar{\mathcal{T}}, \bar{\mathcal{Q}})$ is Σ_1^1 -definable from the parameters $\bar{\mathcal{T}}$ and $\bar{\mathcal{Q}}$ and holds in the model $Lp^{2n-1}(z)$ as witnessed by the branch \bar{b} .

By Σ_1^1 -absoluteness the statement $\phi(\bar{\mathcal{T}}, \bar{\mathcal{Q}})$ also holds in the model $H^{\text{Col}(\omega, \eta)}$ as witnessed by some branch b , where $\eta < H \cap \text{Ord}$ is a large enough ordinal such that $\bar{\mathcal{T}}, \bar{\mathcal{Q}} \in H^{\text{Col}(\omega, \eta)}$ are countable inside $H^{\text{Col}(\omega, \eta)}$.

Since b is uniquely definable from $\bar{\mathcal{T}}$ and the \mathcal{Q} -structure $\bar{\mathcal{Q}}$, and $\bar{\mathcal{T}}, \bar{\mathcal{Q}} \in H$, we have by homogeneity of the forcing $\text{Col}(\omega, \eta)$ that actually $b \in H$. This contradicts the fact that the iteration tree \mathcal{T} witnesses that the premouse $(K^c)^{Lp^{2n-1}(z)}$ is not \mathcal{Q} -structure iterable inside $Lp^{2n-1}(z)$.

Case 2. There is no drop along the branch \bar{b} .

In this case we consider two subcases as follows.

Case 2.1. There is no Woodin cardinal $\bar{\delta}$ in \bar{K} such that $i_b^{\bar{\mathcal{T}}}(\bar{\delta}) = \delta(\bar{\mathcal{T}})$.

In this case we have that $\delta(\bar{\mathcal{T}})$ is not Woodin in $\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$. Thus there exists an initial segment $\mathcal{Q}^* \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$ such that

$$\mathcal{Q}^* \models \text{“}\delta(\bar{\mathcal{T}}) \text{ is a Woodin cardinal”},$$

and definably over \mathcal{Q}^* there exists a witness for the fact that $\delta(\bar{\mathcal{T}})$ is not Woodin (in the sense of Definition 2.7). As $(K^c)^{Lp^{2n-1}(z)}$ is countably iterable in $Lp^{2n-1}(z)$ we can successfully coiterate the premice \mathcal{Q}^* and $\bar{\mathcal{Q}}$. Therefore we have that in fact $\mathcal{Q}^* = \bar{\mathcal{Q}}$ and thus as in Case 1 it follows that $\bar{\mathcal{Q}} \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$ is a \mathcal{Q} -structure for $\bar{\mathcal{T}}$ and we can derive a contradiction from that as above.

Case 2.2. There is a Woodin cardinal $\bar{\delta}$ in \bar{K} such that $i_b^{\bar{\mathcal{T}}}(\bar{\delta}) = \delta(\bar{\mathcal{T}})$.

Let $\delta = \pi(\bar{\delta})$ be the corresponding Woodin cardinal in $(K^c)^{Lp^{2n-1}(z)}$. Then the real z is generic over the model $(K^c)^{Lp^{2n-1}(z)}$ for the extender algebra at the Woodin cardinal δ since all extenders appearing in the K^c -construction in the sense of [MSch04] satisfy the axioms of the extender algebra. In fact by the same argument the model $Lp^{2n-1}(z)|\delta$ is generic over $(K^c)^{Lp^{2n-1}(z)}|(\delta + \omega)$ for the δ -version of the extender algebra \mathbb{Q}_δ (see the proof of Lemma 1.3 in [SchSt09] for a definition of the δ -version of the extender algebra).

Therefore we can perform a maximal \mathcal{P} -construction, which is defined as in [SchSt09], inside $Lp^{2n-1}(z)$ over $(K^c)^{Lp^{2n-1}(z)}|(\delta + \omega)$ to obtain a model \mathcal{P} . We have that $\mathcal{P} \models$ “ δ is Woodin” by the definition of a maximal \mathcal{P} -construction.

Let $\bar{\mathcal{P}}$ be the corresponding result of a maximal \mathcal{P} -construction in H over $\bar{K}|(\bar{\delta} + \omega)$. Moreover let \mathcal{T}^* be the iteration tree obtained by considering the tree $\bar{\mathcal{T}}$ based on $\bar{K}|\bar{\delta}$ as an iteration tree on $\bar{\mathcal{P}} \triangleright \bar{K}|\bar{\delta}$. Let

$$i_{\bar{b}}^{\mathcal{T}^*} : \bar{\mathcal{P}} \rightarrow \mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$$

denote the corresponding iteration embedding, where the branch through \mathcal{T}^* we consider is induced by the branch \bar{b} through $\bar{\mathcal{T}}$ we fixed above, so we call them both \bar{b} . Then we have that

$$i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta}) = i_{\bar{b}}^{\bar{\mathcal{T}}}(\bar{\delta}) = \delta(\bar{\mathcal{T}}).$$

Case 2.2.1. We have that

$$\mathcal{P} \cap \text{Ord} < Lp^{2n-1}(z) \cap \text{Ord}.$$

Then we have in particular that $\bar{\mathcal{P}} \cap \text{Ord} < H \cap \text{Ord}$ and thus $\bar{\mathcal{P}} \in H$.

Consider the coiteration of $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ with $\bar{\mathcal{Q}}$ inside $Lp^{2n-1}(z)$. By the definition of a maximal \mathcal{P} -construction (see [SchSt09]) we have that $\bar{\delta}$ is not definably Woodin over $\bar{\mathcal{P}}$ since $\bar{\mathcal{P}} \cap \text{Ord} < H \cap \text{Ord}$. Since \bar{b} is non-dropping this implies that $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$ is not definably Woodin over $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$. Furthermore we have that

$$\mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \models “i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta}) \text{ is Woodin}”.$$

Concerning the other side of the coiteration we also have that $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta}) = \delta(\bar{\mathcal{T}})$ is a Woodin cardinal in $\bar{\mathcal{Q}}$ but it is not definably Woodin over $\bar{\mathcal{Q}}$.

Since the coiteration of $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ with $\bar{\mathcal{Q}}$ takes place above $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta}) = \delta(\bar{\mathcal{T}})$ we have that it is successful inside $Lp^{2n-1}(z)$ using that $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ inherits the realization strategy for H above $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$ and it follows that in fact

$$\mathcal{M}_{\bar{b}}^{\mathcal{T}^*} = \bar{\mathcal{Q}}.$$

Consider the statement

$$\psi(\mathcal{T}^*, \bar{\mathcal{Q}}) \equiv \text{“there is a cofinal branch } b \text{ through } \mathcal{T}^* \text{ such that } \bar{\mathcal{Q}} = \mathcal{M}_b^{\mathcal{T}^*}\text{”}.$$

This statement $\psi(\mathcal{T}^*, \bar{\mathcal{Q}})$ is Σ_1^1 -definable from the parameters \mathcal{T}^* and $\bar{\mathcal{Q}}$ and holds in the model $Lp^{2n-1}(z)$ as witnessed by the branch \bar{b} . We have that $\bar{\mathcal{T}}, \bar{\mathcal{P}} \in H$ and thus $\mathcal{T}^* \in H$.

Therefore an absoluteness argument exactly as in Case 1 yields that $\psi(\mathcal{T}^*, \bar{\mathcal{Q}})$ holds in $H^{\text{Col}(\omega, \eta)}$, where $\eta < H \cap \text{Ord}$ is an ordinal such that $\mathcal{T}^*, \bar{\mathcal{Q}} \in H^{\text{Col}(\omega, \eta)}$ are countable inside the model $H^{\text{Col}(\omega, \eta)}$. Thus it follows as before

that $\bar{b} \in H$, which contradicts the fact that $\bar{\mathcal{T}}$ witnesses in H that \bar{K} is not \mathcal{Q} -structure iterable.

Case 2.2.2. We have that

$$\mathcal{P} \cap \text{Ord} = Lp^{2n-1}(z) \cap \text{Ord}.$$

Then it follows that $\bar{\mathcal{P}} \cap \text{Ord} = H \cap \text{Ord}$ and Lemma 1.5 in [SchSt09] applied to the maximal \mathcal{P} -construction inside H yields that $\bar{\delta}$ is not definably Woodin over $\bar{\mathcal{P}}$ since we have that $\rho_\omega(H) = \omega$. As \bar{b} is non-dropping, this implies that $i_{\bar{b}}^{\bar{\mathcal{T}}^*}(\bar{\delta})$ is not definably Woodin over $\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*}$.

So as in Case 2.2.1 we can successfully coiterate $\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*}$ and $\bar{\mathcal{Q}}$ inside $Lp^{2n-1}(z)$ and obtain again that

$$\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*} = \bar{\mathcal{Q}}.$$

Then it follows that

$$\bar{\mathcal{Q}} \cap \text{Ord} < H \cap \text{Ord} = \bar{\mathcal{P}} \cap \text{Ord} \leq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*} \cap \text{Ord},$$

where the first inequality holds true since $\bar{\mathcal{Q}} \in H$. This is a contradiction to the fact that $\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}^*} = \bar{\mathcal{Q}}$.

This finishes the proof of Subclaim 5. □

Working in $Lp^{2n-1}(z)$ we consider the coiteration of $K^{M_x}|\omega_2^{Lp^{2n-1}(z)}$ with $(K^c|\omega_2)^{Lp^{2n-1}(z)}$. From what we proved so far it easily follows that this coiteration is successful as shown in the next subclaim.

Subclaim 6. *The coiteration of $K^{M_x}|\omega_2^{Lp^{2n-1}(z)}$ with $(K^c|\omega_2)^{Lp^{2n-1}(z)}$ in $Lp^{2n-1}(z)$ is successful.*

Proof. Write $W = Lp^{2n-1}(z)$. Then we can successfully coiterate the premice $K^{M_x}|\omega_2^W$ and $K^c|\omega_2^W$ inside the model W since $K^{M_x}|\omega_2^W = K^{M_x}|(\gamma^+)^{M_x}$ is fully iterable in W by Subclaim 3 and $K^c|\omega_2^W$ is \mathcal{Q} -structure iterable in W by Subclaim 5. In particular the $K^{M_x}|\omega_2^W$ -side of the coiteration provides \mathcal{Q} -structures for the $K^c|\omega_2^W$ -side and therefore the coiteration is successful. □

In what follows we want to argue that the $(K^c|\omega_2)^{Lp^{2n-1}(z)}$ -side cannot lose this coiteration. For that we want to use the following lemma, which we can prove similar as Theorem 3.8 in [MSch04]. As the version we aim to use is slightly stronger than what is shown in [MSch04], we will sketch a proof of this lemma.

Lemma 3.31. *Let $\kappa \geq \omega_2$ be a regular cardinal such that κ is inaccessible in K^c (constructed in the sense of [MSch04]). Then $K^c|\kappa$ is universal with respect to every premouse M with $M \cap \text{Ord} = \kappa$ to which it can be successfully coiterated.*

In [MSch04] the universality of the premouse $K^c|\kappa$ is only proved with respect to smaller premice, i.e. premice M such that $M \cap \text{Ord} < \kappa$, to which $K^c|\kappa$ can be successfully coiterated. As shown below, their argument can easily be modified to yield Lemma 3.31.

Proof of Lemma 3.31. Let $\kappa \geq \omega_2$ be a regular cardinal such that κ is inaccessible in K^c and write $N = K^c|\kappa$.

Analogous to the notation in [MSch04] we say that M *iterates past* N iff M is a premouse with $M \cap \text{Ord} = \kappa$ and there are iteration trees \mathcal{T} on M and \mathcal{U} on N of length $\lambda + 1$ arising from a successful comparison such that there is no drop along $[0, \lambda]_{\mathcal{U}}$ and $\mathcal{M}_\lambda^{\mathcal{U}} \triangleleft \mathcal{M}_\lambda^{\mathcal{T}}$ or $\mathcal{M}_\lambda^{\mathcal{U}} = \mathcal{M}_\lambda^{\mathcal{T}}$ and we in addition have that there is a drop along $[0, \lambda]_{\mathcal{T}}$.

Assume toward a contradiction that there is a premouse M which iterates past N and let \mathcal{T} and \mathcal{U} be iteration trees of length $\lambda + 1$ on M and N respectively witnessing this. Let $i_{0\lambda}^{\mathcal{U}} : N \rightarrow \mathcal{M}_\lambda^{\mathcal{U}}$ denote the corresponding iteration embedding on the N -side, which exists as there is no drop on the main branch through \mathcal{U} .

We distinguish the following cases.

Case 1. We have that for some $\xi < \kappa$,

$$i_{0\lambda}^{\mathcal{U}} \text{'' } \kappa \subset \kappa \text{ and } i_{\beta\lambda}^{\mathcal{T}}(\xi) \geq \kappa$$

for some ordinal $\beta < \lambda$ such that the iteration embedding $i_{\beta\lambda}^{\mathcal{T}}$ is defined.

In this case we can derive a contradiction as in Section 3 in [MSch04] because we can prove that the consequences of Lemma 3.5 in [MSch04] also hold in this setting.

We assume that the reader is familiar with the argument for Lemma 3.5 in [MSch04] and we use the notation from there. So we let

$$X \prec H_{\kappa^+}$$

be such that $|X| < \kappa$, $X \cap \kappa \in \kappa$, $\{M, N, \mathcal{T}, \mathcal{U}, \beta, \xi\} \subset X$, and $X \cap \kappa \in (0, \kappa)_{\mathcal{T}} \cap (0, \kappa)_{\mathcal{U}}$, as in this case $\lambda = \kappa$. We write $\alpha = X \cap \kappa$. Let

$$\pi : \bar{H} \cong X \prec H_{\kappa^+}$$

be such that \bar{H} is transitive. Then we have that $\alpha = \text{crit}(\pi)$ and $\pi(\alpha) = \kappa$. Let $\bar{\mathcal{T}}, \bar{\mathcal{U}} \in \bar{H}$ be such that $\pi(\bar{\mathcal{T}}, \bar{\mathcal{U}}) = (\mathcal{T}, \mathcal{U})$.

As in the proof of Lemma 3.5 in [MSch04] we aim to show that $\mathcal{P}(\alpha) \cap N \subset \bar{H}$. Exactly as in [MSch04] we get that

$$\mathcal{M}_\alpha^{\mathcal{T}} \upharpoonright (\alpha^+) \mathcal{M}_\alpha^{\mathcal{T}} = \mathcal{M}_\alpha^{\mathcal{U}} \upharpoonright (\alpha^+) \mathcal{M}_\alpha^{\mathcal{U}}.$$

The case assumption that $i_{\beta\kappa}^{\mathcal{T}}(\xi) \geq \kappa$ for ordinals $\beta, \xi < \kappa$ implies that there are ordinals $\bar{\beta}, \bar{\xi} < \alpha$ such that $i_{\bar{\beta}\alpha}^{\bar{\mathcal{T}}}(\bar{\xi}) \geq \alpha$. As $i_{\bar{\beta}\alpha}^{\bar{\mathcal{T}}} \upharpoonright \alpha = i_{\bar{\beta}\alpha}^{\mathcal{T}} \upharpoonright \alpha$ and $\mathcal{M}_\alpha^{\bar{\mathcal{T}}}|_\alpha = \mathcal{M}_\alpha^{\mathcal{T}}|_\alpha$, this yields that $\mathcal{P}(\alpha) \cap \mathcal{M}_\alpha^{\mathcal{T}} = \mathcal{P}(\alpha) \cap \mathcal{M}_\alpha^{\bar{\mathcal{T}}}$. Therefore we

have that

$$\mathcal{P}(\alpha) \cap \mathcal{M}_\alpha^{\mathcal{U}} = \mathcal{P}(\alpha) \cap \mathcal{M}_\alpha^{\mathcal{T}} = \mathcal{P}(\alpha) \cap \mathcal{M}_\alpha^{\bar{H}} \in \bar{H}.$$

Moreover we are assuming that $i_{0\kappa}^{\mathcal{U}} \restriction \kappa \subset \kappa$ and we have

$$i_{0\alpha}^{\mathcal{U}} \restriction \alpha = i_{0\alpha}^{\bar{H}} \restriction \alpha \in \bar{H}.$$

Therefore we can again argue exactly as in the proof of Lemma 3.5 in [MSch04] to get that $\mathcal{P}(\alpha) \cap N \subset \bar{H}$. Following [MSch04] this now yields a contradiction to the assumption that M iterates past N .

Case 2. We have that

$$i_{0\lambda}^{\mathcal{U}} \restriction \kappa \subset \kappa \text{ and there are no } \xi, \beta \text{ such that } i_{\beta\lambda}^{\mathcal{T}}(\xi) \geq \kappa,$$

where $\xi < \kappa$ and $\beta < \lambda$.

By assumption we have that κ is inaccessible in N . Assume first that M has a largest cardinal η . In this case we have that there are no cardinals between the image of η under the iteration embedding and κ in $\mathcal{M}_\lambda^{\mathcal{T}}$ but there are cardinals between the image of η under the iteration embedding and κ in $\mathcal{M}_\lambda^{\mathcal{U}}$. This contradicts the fact that \mathcal{T} and \mathcal{U} were obtained by a successful comparison of M and N with $\mathcal{M}_\lambda^{\mathcal{U}} \trianglelefteq \mathcal{M}_\lambda^{\mathcal{T}}$.

Now assume that $M \models \text{ZFC}$. Then the case assumption implies that in particular $\mathcal{M}_\lambda^{\mathcal{T}} \cap \text{Ord} \leq \kappa$. This contradicts the assumption that M iterates past N .

Case 3. We have that for some $\zeta < \kappa$,

$$i_{0\lambda}^{\mathcal{U}}(\zeta) \geq \kappa \text{ and there are no } \xi, \beta \text{ such that } i_{\beta\lambda}^{\mathcal{T}}(\xi) \geq \kappa,$$

where $\xi < \kappa$ and $\beta < \lambda$.

This again easily contradicts the assumption that M iterates past N .

Case 4. We have that for some $\zeta < \kappa$ and for some $\xi < \kappa$,

$$i_{0\lambda}^{\mathcal{U}}(\zeta) \geq \kappa \text{ and } i_{\beta\lambda}^{\mathcal{T}}(\xi) \geq \kappa$$

for some ordinal $\beta < \lambda$ such that the iteration embedding $i_{\beta\lambda}^{\mathcal{T}}$ is defined.

In this case we have that $\lambda = \kappa$. Let

$$X \prec H_\theta$$

for some large enough ordinal θ be such that $|X| = \kappa$ and $\{M, N, \mathcal{T}, \mathcal{U}\} \subset X$. Let

$$\pi : \bar{H} \cong X \prec H_\theta$$

be such that \bar{H} is transitive.

Then a reflection argument as in the proof of the Comparison Lemma (see Theorem 3.11 in [St10]) yields that there is an ordinal $\gamma < \kappa$ such that the embeddings $i_{\gamma\kappa}^{\mathcal{T}}$ and $i_{\gamma\kappa}^{\mathcal{U}}$ agree. This implies that there are extenders $E_\alpha^{\mathcal{T}}$

used in \mathcal{T} at stage α and $E_{\alpha'}^{\mathcal{U}}$ used in \mathcal{U} at stage α' which are compatible. Again as in the proof of Theorem 3.11 in [St10] this yields a contradiction. This finishes the proof of Lemma 3.31. \square

Now we can use Lemma 3.31 to prove the following subclaim.

Subclaim 7. $\omega_2^{Lp^{2n-1}(z)}$ is a successor cardinal in $(K^c)^{Lp^{2n-1}(z)}$.

Proof. Work in $W = Lp^{2n-1}(z)$ and assume the converse. That means we are assuming that ω_2^W is inaccessible in K^c .

As above consider the successful coiteration of $K^{M_x}|\omega_2^W$ with $K^c|\omega_2^W$ and let \mathcal{T} and \mathcal{U} be the resulting trees on $K^{M_x}|\omega_2^W$ and $K^c|\omega_2^W$ respectively of length $\lambda + 1$ for some ordinal λ . Since we assume that ω_2^W is inaccessible in K^c , it follows by Lemma 3.31 that $K^c|\omega_2^W$ is universal in W for the coiteration with preimage of height $\leq \omega_2^W$. Therefore the $K^c|\omega_2^W$ -side has to win the comparison. That means we have that $\mathcal{M}_\lambda^{\mathcal{T}} \trianglelefteq \mathcal{M}_\lambda^{\mathcal{U}}$ and there is no drop on the main branch through $\mathcal{M}_\lambda^{\mathcal{T}}$. In particular the iteration embedding

$$i_{0\lambda}^{\mathcal{T}} : K^{M_x}|\omega_2^W \rightarrow \mathcal{M}_\lambda^{\mathcal{T}}$$

exists. Now we distinguish the following cases.

Case 1. We have that

$$i_{0\lambda}^{\mathcal{T}} \text{ `` } \omega_2^W \subset \omega_2^W.$$

This means in particular that $(\gamma^+)^{K^{M_x}} = (\gamma^+)^{M_x} = \omega_2^W$ stays a successor cardinal in the model $\mathcal{M}_\lambda^{\mathcal{T}}$. So say that we have $(\eta^+)^{\mathcal{M}_\lambda^{\mathcal{T}}} = \omega_2^W$ for some cardinal $\eta < \omega_2^W$ in $\mathcal{M}_\lambda^{\mathcal{T}}$. In particular this means that there are no cardinals between η and ω_2^W in $\mathcal{M}_\lambda^{\mathcal{T}}$. But by assumption we have that ω_2^W is inaccessible in K^c and thus also in $\mathcal{M}_\lambda^{\mathcal{U}}$. In particular there are cardinals between η and ω_2^W in $\mathcal{M}_\lambda^{\mathcal{U}}$. This contradicts the fact that $\mathcal{M}_\lambda^{\mathcal{T}}$ and $\mathcal{M}_\lambda^{\mathcal{U}}$ were obtained by a successful comparison with $\mathcal{M}_\lambda^{\mathcal{T}} \trianglelefteq \mathcal{M}_\lambda^{\mathcal{U}}$.

Case 2. We have that

$$i_{0\lambda}^{\mathcal{T}} \text{ `` } \omega_2^W \not\subset \omega_2^W.$$

In this case we distinguish two subcases as follows.

Case 2.1. We have that

$$\sup i_{0\lambda}^{\mathcal{T}} \text{ `` } \gamma < \omega_2^W.$$

In this case we have that

$$(i_{0\lambda}^{\mathcal{T}}(\gamma^+))^{\mathcal{M}_\lambda^{\mathcal{T}}} = \omega_2^W,$$

because we are also assuming that $i_{0\lambda}^{\mathcal{T}} \text{ `` } \omega_2^W \not\subset \omega_2^W$.

So in particular we again have that ω_2^W is a successor cardinal in the model $\mathcal{M}_\lambda^\mathcal{T}$ and so there are no cardinals between $i_{0\lambda}^\mathcal{T}(\gamma) < \omega_2^W$ and ω_2^W in $\mathcal{M}_\lambda^\mathcal{T}$. From this we can derive the same contradiction as in Case 1, because ω_2^W is inaccessible in K^c .

Case 2.2. We have that

$$\exists \eta < \omega_2^W \text{ such that } i_{0\lambda}^\mathcal{T}(\eta) \geq \omega_2^W.$$

Let $\alpha < \lambda$ be the least ordinal such that the iteration embedding $i_{\alpha\lambda}^\mathcal{U}$ is defined. That means the last drop on the main branch in \mathcal{U} is at stage α . Since $K^{M_x}|\omega_2^W$ has height ω_2^W we have by universality of $K^c|\omega_2^W$ in W (see Lemma 3.31) that the case assumption implies that there exists an ordinal $\nu < \omega_2^W$ such that

$$i_{\alpha\lambda}^\mathcal{U}(\nu) \geq \omega_2^W.$$

Moreover in this case we have in fact $\lambda = \omega_2^W$.

Let $X \prec W|\theta$ for some large ordinal θ be such that $\mathcal{T}, \mathcal{U} \in X$ and $|X| = \omega_2^W$. Moreover let H be the Mostowski collapse of X and let $\pi : H \rightarrow W|\theta$ be the uncollapse map. Then a reflection argument as in the proof of the Comparison Lemma (see Theorem 3.11 in [St10]) yields that there is an ordinal $\xi < \lambda$ such that the embeddings $i_{\xi\lambda}^\mathcal{T}$ and $i_{\xi\lambda}^\mathcal{U}$ agree. This implies that there are extenders $E_\beta^\mathcal{T}$ used in \mathcal{T} at stage β and $E_{\beta'}^\mathcal{U}$ used in \mathcal{U} at stage β' which are compatible. Again as in the proof of Theorem 3.11 in [St10] this yields a contradiction.

This finishes the proof of Subclaim 7. □

Recall that we have

$$\text{HOD}^{Lp^{2n-1}(z)} \models \text{“}\omega_2^{Lp^{2n-1}(z)} \text{ is inaccessible”}$$

by Theorem 3.27 as $z \geq_T x$. Since we have that

$$(K^c)^{Lp^{2n-1}(z)} \subset \text{HOD}^{Lp^{2n-1}(z)}$$

by Subclaim 4, this contradicts Subclaim 7 and thereby finishes the proof of Claim 3. □

Work in V now and let $x \in {}^\omega\omega$ be arbitrary in the cone of reals from Theorem 3.27. Then by Claim 3 we have that

$$M_x \models \text{“}(M_{2n}^\#)^{M_x} \text{ is } \omega_1\text{-iterable”}.$$

Hence

$$M_x \models \text{“}(M_{2n}^\#)^{M_x} \text{ is } \Pi_{2n+2}^1\text{-iterable”}.$$

Since M_x is Σ_{2n+2}^1 -correct in V we have that

$$V \models \text{“}(M_{2n}^\#)^{M_x} \text{ is } \Pi_{2n+2}^1\text{-iterable”}.$$

By Σ_{2n+2}^1 -correctness in V again we have for every real $y \geq_T x$ such that in particular $(M_{2n}^\#)^{M_x} \in M_y$ that

$$M_y \models \text{“}(M_{2n}^\#)^{M_x} \text{ is } \Pi_{2n+2}^1\text{-iterable”}.$$

Consider the comparison of the premice $(M_{2n}^\#)^{M_x}$ and $(M_{2n}^\#)^{M_y}$ inside the model M_y . This comparison is successful by Lemma 2.12 for all reals $y \geq_T x$ as above, since $(M_{2n}^\#)^{M_y}$ is ω_1 -iterable in M_y and $(M_{2n}^\#)^{M_x}$ is Π_{2n+2}^1 -iterable in M_y . Moreover both premice are ω -sound and we have that $\rho_\omega((M_{2n}^\#)^{M_x}) = \rho_\omega((M_{2n}^\#)^{M_y}) = \omega$. Thus the premice $(M_{2n}^\#)^{M_x}$ and $(M_{2n}^\#)^{M_y}$ are in fact equal.

Therefore we have that all premice $(M_{2n}^\#)^{M_x}$ for $x \in {}^\omega\omega$ in the cone of reals from Theorem 3.27 are equal in V . Call this unique premouse $M_{2n}^\#$.

We now finally show that this premouse $M_{2n}^\#$ is ω_1 -iterable in V via the \mathcal{Q} -structure iteration strategy (see Definition 2.5). So let \mathcal{T} be an iteration tree on $M_{2n}^\#$ in V of limit length $< \omega_1^V$ according to the \mathcal{Q} -structure iteration strategy. Pick $z \in {}^\omega\omega$ such that $M_{2n}^\#$ and \mathcal{T} are in M_z and $\text{lh}(\mathcal{T}) < \omega_1^{M_z}$.

Since M_z is Σ_{2n+2}^1 -correct in V we have that \mathcal{T} is according to the \mathcal{Q} -structure iteration strategy in M_z , because $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ is $2n$ -small above $\delta(\mathcal{T} \upharpoonright \lambda)$ for all limit ordinals $\lambda < \text{lh}(\mathcal{T})$ and therefore Π_{2n+1}^1 -iterability above $\delta(\mathcal{T} \upharpoonright \lambda)$ is enough for $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ to determine a unique cofinal well-founded branch b through \mathcal{T} by Lemma 2.13. Moreover we have that $(M_{2n}^\#)^{M_z} = M_{2n}^\#$ and therefore

$$M_z \models \text{“}M_{2n}^\# \text{ is } \omega_1\text{-iterable”}.$$

So in M_z there exists a cofinal well-founded branch b through \mathcal{T} , which is determined by \mathcal{Q} -structures $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ which are ω_1 -iterable above $\delta(\mathcal{T} \upharpoonright \lambda)$ and therefore also Π_{2n+1}^1 -iterable above $\delta(\mathcal{T} \upharpoonright \lambda)$ in M_z for all limit ordinals $\lambda \leq \text{lh}(\mathcal{T})$. That means we in particular have that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$. Since M_z is Σ_{2n+2}^1 -correct in V , it follows as above that b is also the unique cofinal well-founded branch in V which is determined by the same \mathcal{Q} -structures as in M_z . Therefore

$$V \models \text{“}M_{2n}^\# \text{ exists and is } \omega_1\text{-iterable”}$$

and we finished the proof of Theorem 3.20. \square

4. CONCLUSION

By the results proved in Sections 3.5 and 3.6 we have that the following theorem which is due to Itay Neeman and the third author and was announced in Section 3.3 holds true.

Theorem 3.13. *Let $n \geq 1$ and assume there is no Σ_{n+2}^1 -definable ω_1 -sequence of pairwise distinct reals. Then the following are equivalent.*

- (1) Π_n^1 determinacy and Π_{n+1}^1 determinacy,

- (2) for all $x \in {}^\omega\omega$, $M_{n-1}^\#(x)$ exists and is ω_1 -iterable, and $M_n^\#$ exists and is ω_1 -iterable,
- (3) $M_n^\#$ exists and is ω_1 -iterable.

Proof. This follows from Theorems 3.16 and 3.20 together with Theorem 2.14 in [Ne02]. □

Moreover Theorems 3.16 and 3.20 together with Lemma 3.15 immediately imply the following main theorem due to the third author.

Theorem 2.1. *Let $n \geq 1$ and assume Π_{n+1}^1 determinacy holds. Then $M_n^\#(x)$ exists and is ω_1 -iterable for all $x \in {}^\omega\omega$.*

4.1. Applications. From these results we can now obtain a boldface version of the Determinacy Transfer Theorem as in Theorem 3.17 for all projective levels Π_{n+1}^1 of determinacy. The lightface version of the Determinacy Transfer Theorem for even levels of projective determinacy Π_{2n}^1 (see Theorem 3.17) is used in the proof of Theorem 3.16, and therefore in the proof of Theorem 2.1 for odd levels n .

Corollary 4.1 (Determinacy Transfer Theorem). *Let $n \geq 1$. Then Π_{n+1}^1 determinacy is equivalent to $\wp^{(n)}(< \omega^2 - \Pi_1^1)$ determinacy.*

Proof. By Theorem 1.10 in [KW08] we have for the even levels that

$$\text{Det}(\Pi_{2n}^1) \leftrightarrow \text{Det}(\Delta_{2n}^1) \leftrightarrow \text{Det}(\wp^{(2n-1)}(< \omega^2 - \Pi_1^1)).$$

Here the first equivalence is due to Martin (see [Ma73]) and proven in Theorem 5.1 in [KS85]. The second equivalence due to Kechris and the third author can be proven using purely descriptive set theoretic methods (see Theorem 1.10 in [KW08]).

The results in this paper (using this version of the Determinacy Transfer Theorem for even levels, see Theorem 3.17) together with results due to Itay Neeman yield the Determinacy Transfer Theorem for all levels n as follows.

By basic facts about the game quantifier “ \wp ” we have that

$$\text{Det}(\wp^{(n)}(< \omega^2 - \Pi_1^1))$$

implies Π_{n+1}^1 determinacy.

For the other direction assume that Π_{n+1}^1 determinacy holds. Then Theorem 2.1 yields that the premouse $M_n^\#(x)$ exists and is ω_1 -iterable for all $x \in {}^\omega\omega$. This implies that

$$\text{Det}(\wp^{(n)}(< \omega^2 - \Pi_1^1))$$

holds true by Theorem 2.5 in [Ne95]. □

4.2. Open problems. We close this paper with the following open problem, which is the lightface version of Theorem 2.1.

Conjecture. *Let $n > 1$ and assume that Π_n^1 determinacy and Π_{n+1}^1 determinacy hold. Then $M_n^\#$ exists and is ω_1 -iterable.*

This conjecture holds true for $n = 0$ which is due to L. Harrington (see [Ha78]) and for $n = 1$ which is due to the third author (see Corollary 4.17 in [StW16]), but it is open for $n > 1$.

REFERENCES

- [Da64] M. Davis. Infinite games of perfect information. In *Advances in Game Theory*, pages 85–101. Princeton University Press, 1964.
- [FNS10] G. Fuchs, I. Neeman, and R. Schindler. A Criterion for Coarse Iterability. *Archive for Mathematical Logic*, 49, 2010.
- [FSS14] S. D. Friedman, R. Schindler, and D. Schritterser. Coding over core models. In S. Geschke, B. Löwe, and P. Schlicht, editors, *Infinity, Computability, and Metamathematics, Festschrift celebrating the 60th birthdays of Peter Koepke and Philip Welch*, pages 167–182. 2014.
- [GS53] D. Gale and Stewart F. M. Infinite games with perfect information. In H. W. Kuhn and A. W. Tucker, editors, *Contributions to the Theory of Games*, volume 2, pages 245–266. Princeton University Press, 1953.
- [Ha78] L. Harrington. Analytic Determinacy and $0^\#$. *Journal of Symbolic Logic*, 43:685–693, 1978.
- [JS13] R. B. Jensen and J. R. Steel. K without the measurable. *J. Symb. Log.*, 78(3):708–734, 2013.
- [Je03] R. B. Jensen. Robust extenders. *Handwritten notes*, 2003.
- [Je72] R. B. Jensen. The fine structure of the constructible hierarchy. *Ann. Math. Logic*, 4:229–308, 1972.
- [Je97] R. B. Jensen. A new fine structure for higher core models. *Circulated manuscript*, 1997.
- [KM08] A. S. Kechris and Y. N. Moschovakis. Notes on the theory of scales. In A. S. Kechris, B. Löwe, and J. R. Steel, editors, *Games, Scales and Suslin Cardinals, The Cabal Seminar, Volume I*. Cambridge University Press, 2008.
- [KS85] A. Kechris and R. Solovay. On the Relative Consistency Strength of Determinacy Hypotheses. *Transactions of the American Mathematical Society*, 290(1), 1985.
- [KW08] A. S. Kechris and W. H. Woodin. The Equivalence of partition properties and determinacy. In A. S. Kechris, B. Löwe, and J. R. Steel, editors, *Games, Scales and Suslin Cardinals, The Cabal Seminar, Volume I*. Cambridge University Press, 2008.

- [KW10] P. Koellner and W. H. Woodin. Large Cardinals from Determinacy. In M. Foreman and A. Kanamori, editors, *Handbook of Set Theory*. Springer, 2010.
- [Ka08] A. Kanamori. *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings*. Springer Monographs in Mathematics. Springer, 2008.
- [Kei62] H. Jerome Keisler. Some applications of the theory of models to set theory. *Proc. Int. Cong. of Logic, Methodology, and Philosophy of Science*, pages 80–85, 1962.
- [La04] P. B. Larson. *The Stationary Tower: Notes on a Course by W. Hugh Woodin*, volume 32 of *University Lecture Series*. AMS, 2004.
- [MS94] W. J. Mitchell and J. R. Steel. *Fine structure and iteration trees*. Lecture notes in logic. Springer-Verlag, Berlin, New York, 1994.
- [MSch04] W. J. Mitchell and R. Schindler. A universal extender model without large cardinals in V . *Journal of Symbolic Logic*, 69(2):371–386, 2004.
- [MSch95] W. J. Mitchell and E. Schimmerling. Covering without countable closure. *Mathematical Research Letters*, 2(5):595–609, 1995.
- [Ma08] D. A. Martin. The largest countable this, that, and the other. In A. S. Kechris, B. Löwe, and J. R. Steel, editors, *Games, Scales and Suslin Cardinals, The Cabal Seminar, Volume I*. Cambridge University Press, 2008.
- [Ma70] D. A. Martin. Measurable cardinals and analytic games. *Fundamenta Mathematicae*, 66:287–291, 1970.
- [Ma73] D. A. Martin. Δ_{2n}^1 determinacy implies Σ_{2n}^1 determinacy. *Mimeographed notes*, 1973.
- [Ma75] D. A. Martin. Borel Determinacy. *Annals of Mathematics*, 102(2):363–371, 1975.
- [MaSt89] D. A. Martin and J. R. Steel. A proof of projective determinacy. *Journal of the American Mathematical Society*, 2:71–125, 1989.
- [MaSt94] D. A. Martin and J. R. Steel. Iteration trees. *Journal of the American Mathematical Society*, 7(1):1–73, 1994.
- [Mo09] Y. N. Moschovakis. *Descriptive set theory, second edition*, volume 155 of *Mathematical Surveys and Monographs*. AMS, 2009.
- [Mo71] Y. N. Moschovakis. Uniformization in a playful universe. *Bulletin of the American Mathematical Society*, 77:731–736, 1971.
- [MySw64] J. Mycielski and S. Swierczkowski. On the Lebesgue measurability and the axiom of determinateness. *Fundamenta Mathematicae*, 54:67–71, 1964.
- [Ne02] I. Neeman. Optimal Proofs of Determinacy II. *Journal of Mathematical Logic*, 2(2):227–258, 11 2002.
- [Ne10] I. Neeman. Determinacy in $L(\mathbb{R})$. In M. Foreman and A. Kanamori, editors, *Handbook of Set Theory*. Springer, 2010.
- [Ne95] I. Neeman. Optimal Proofs of Determinacy. *Bulletin of Symbolic Logic*, 1(3):327–339, 09 1995.

- [Sa13] G. Sargsyan. On the prewellorderings associated with directed systems of mice. *J. Symb. Log.*, 78(3):735–763, 2013.
- [Sc61] Dana S. Scott. Measurable cardinals and constructible sets. *Bulletin de l'Académie Polonaise des Sciences*, 9:521–524, 1961.
- [Sch00] R. Schindler. Coding into K by reasonable forcing. *Transactions of the American Mathematical Society*, 353:479–489, 2000.
- [Sch10] E. Schimmerling. A Core Model Toolbox and Guide. In M. Foreman and A. Kanamori, editors, *Handbook of Set Theory*. Springer, 2010.
- [Sch14] R. Schindler. *Set Theory*. Universitext. Springer-Verlag, 2014.
- [SchSt] R. Schindler and J. R. Steel. *The core model induction*.
- [SchSt09] R. Schindler and J. R. Steel. The self-iterability of $L[E]$. *J. Symb. Log.*, 74(3):751–779, 2009.
- [SchStZe02] R. Schindler, J. R. Steel, and M. Zeman. Deconstructing inner model theory. *Journal of Symbolic Logic*, 67:721–736, 2002.
- [SchT] F. Schlutzenberg and N. Trang. Scales in hybrid mice over \mathbb{R} . *Submitted*, 2016.
- [SchZe10] R. Schindler and M. Zeman. Fine Structure. In M. Foreman and A. Kanamori, editors, *Handbook of Set Theory*. Springer, 2010.
- [So69] R. M. Solovay. The cardinality of Σ_2^1 sets of reals. In J. J. Bulloff, T. C. Holyoke, and S. W. Hahn, editors, *Symposium Papers Commemorating the Sixtieth Birthday of Kurt Gödel*, pages 58–73. Springer Berlin Heidelberg, 1969.
- [St10] J. R. Steel. An Outline of Inner Model Theory. In M. Foreman and A. Kanamori, editors, *Handbook of Set Theory*. Springer, 2010.
- [St93] J. R. Steel. Inner models with many Woodin cardinals. *Annals of Pure and Applied Logic*, 65:185–209, 1993.
- [St95] J. R. Steel. Projectively well-ordered inner models. *Annals of Pure and Applied Logic*, 74, 1995.
- [St96] J. R. Steel. *The core model iterability problem*, volume 8 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, New York, 1996.
- [StW16] J. R. Steel and W. H. Woodin. HOD as a core model. In A. S. Kechris, B. Löwe, and J. R. Steel, editors, *Ordinal definability and recursion theory, The Cabal Seminar, Volume III*. Cambridge University Press, 2016.
- [Wo55] Philip Wolfe. The strict determinateness of certain infinite games. *Pacific J. Math.*, 5:841–847, 1955.
- [Wo82] W. H. Woodin. On the Consistency Strength of Projective Uniformization. In J. Stern, editor, *Proceedings of the Herbrand symposium. Logic Colloquium '81*, pages 365–383. North-Holland, 1982.