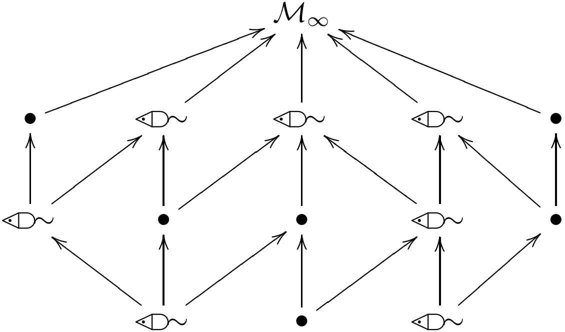
HOD in $M_n(x, g)$

Sandra Uhlenbrock

January 25th-30th, 2017

work in progress with Grigor Sargsyan

Arctic Set Theory Workshop 3, Kilpisjärvi, Finland



Some like it HOD

UC Irvine, July 18 – 29, 2016

Drawing by Martin Zeman.

Motivation

- Want to understand HOD^M for various inner models M like $L(\mathbb{R}), L[x]$ or $M_n(x)$ (assuming determinacy).
- Test question: Is HOD^M a model of GCH?
- Goal: Show that HOD^M is a core model (i.e. a fine structural model).
- This would imply that we have $\mathrm{GCH}, \Diamond, \Box, \dots$ in HOD^M.

What is known about $\mathrm{HOD}^{L(\mathbb{R})}$

Assume $\mathrm{AD}^{L(\mathbb{R})}$.

- (Becker, 1980) $\mathrm{HOD}^{L(\mathbb{R})} \models \mathrm{GCH}_\alpha$ for all $\alpha < \omega_1^V$.
- (Steel, Woodin, 1993) $\mathrm{HOD}^{L(\mathbb{R})} \cap \mathbb{R} = M_\omega \cap \mathbb{R}$.
- (Steel, Woodin, 1993)

  $$\mathrm{HOD}^{L(\mathbb{R})} \cap \mathcal{P}(\omega_1^V) = N \cap \mathcal{P}(\omega_1^V),$$

  where N is the ω_1^V-th iterate of M_ω by it's least measure.
- (Steel, 1995)

  $$\mathrm{HOD}^{L(\mathbb{R})} \cap V_{(\boldsymbol{\delta}_1^2)^{L(\mathbb{R})}} = M_\infty \cap V_{(\boldsymbol{\delta}_1^2)^{L(\mathbb{R})}},$$

  where M_∞ is a direct limit of iterates of M_ω, and $(\boldsymbol{\delta}_1^2)^{L(\mathbb{R})} = \sup\{\alpha \mid \exists f (f : \mathbb{R} \to \alpha$ and f is surjective and $\boldsymbol{\Delta}_1^{L(\mathbb{R})})\}$.
- (Woodin, $\approx$1996)

  $$\mathrm{HOD}^{L(\mathbb{R})} = L[M_\infty, \Lambda],$$

  where Λ is a partial iteration strategy for M_∞.

What is known about $\mathrm{HOD}^{L[x]}$

… very little.

Question

Assume Δ^1_2-determinacy. Do we have

$$\mathrm{HOD}^{L[x]} \models \mathrm{GCH}$$

for a Turing cone of reals x?

What we can do is (under the right determinacy assumption) analyze $\mathrm{HOD}^{L[x][G]}$ for a Turing cone of reals x, where

- G is $\mathrm{Col}(\omega, {<}\kappa_x)$-generic over $L[x]$, and
- $\kappa_x =$ least inaccessible cardinal in $L[x]$.

$\mathrm{HOD}^{L[x,G]}$ as a core model

For every real x let κ_x denote the least inaccessible cardinal in $L[x]$.

Theorem (Woodin, 90's)

Assume Δ^1_2-determinacy. For a Turing cone of x,

$$\mathrm{HOD}^{L[x,G]} = L[M_\infty, \Lambda],$$

where G is $\mathrm{Col}(\omega, {<}\kappa_x)$-generic over $L[x]$, M_∞ is a direct limit of mice, and Λ is a partial iteration strategy for M_∞.

HOD in $M_n(x, g)$

Assume $\mathbf{\Pi}^1_{n+2}$-determinacy.

Goal: Generalize this analysis to $\mathrm{HOD}^{M_n(x)[g]}$ for a Turing cone of reals x, where

- $M_n(x)$ denotes the least proper class iterable premouse with n Woodin cardinals,
- g is $\mathrm{Col}(\omega, {<}\kappa_x)$-generic over $M_n(x)$, and
- $\kappa_x < \delta_0^{M_n(x)}$ is an inaccessible strong cutpoint cardinal of $M_n(x)$ such that κ_x is a limit of strong cutpoint cardinals in $M_n(x)$.

The idea of the proof (very sketchy!)

Let x be a real such that $M_{n+1}^{\#} \in M_n(x)$.

- Define a direct limit system of iterates of $M_{n+1}|(\delta_0^{+\omega})^{M_{n+1}}$ which have a Woodin cardinal that is countable in $M_n(x)[g]$ together with iteration embeddings, call the direct limit M_∞^+.
- M_∞^+ is well-founded as M_{n+1} is sufficiently iterable.
- Define an internal direct limit system of suitable strongly s-iterable premice in $M_n(x)[g]$ and call its direct limit M_∞.
- Sargsyan: $M_\infty = M_\infty^+$, so in particular M_∞ is well-founded.
- Sargsyan: $\delta^{M_\infty} = (\kappa_x^+)^{M_n(x)}$.
- By definability of the internal direct limit system we have that

$$M_\infty \subseteq \mathrm{HOD}^{M_n(x)[g]}.$$

The idea of the proof (very sketchy!)

Let κ_∞ be the least inaccessible cardinal of M_∞ strictly above δ_∞.

- $M_\infty[H]$ for a $\mathrm{Col}(\omega, {<}\kappa_\infty)$-generic H is the *derived model* of M_∞.
- Use the derived model as a surrogate for $M_n(x)[g]$ to compute $\mathrm{HOD}^{M_n(x)[g]}$.

Lemma (Derived model resemblance, Woodin)

The derived model $M_\infty[H]$ (adding the theory of M_n on top) is elementary equivalent to $M_n(x)[g]$.

- Therefore $M_\infty[H]$ has its own version of the direct limit system, call the direct limit model $M_\infty^* = (M_\infty)^{M_\infty[H]}$.
- M_∞ shows up in this direct limit system, let $\pi_\infty : M_\infty \to M_\infty^*$ be the corresponding map.
- In fact, $\pi_\infty \restriction \alpha \in M_\infty$ for all $\alpha < \delta$.

$\mathrm{HOD}^{M_n(x,g)}$

Using this we can show:

Theorem

$$\mathrm{HOD}^{M_n(x)[g]} \cap V_{\delta_\infty} = M_\infty \cap V_{\delta_\infty}.$$

Moreover we are optimistic to show:

Lemma

For some $M_n(x)[g]$-definable set $A \subseteq \omega_2^{M_n(x)[g]}$ we have that

$$\mathrm{HOD}^{M_n(x)[g]} = M_n(A).$$

This should then give that

$$\mathrm{HOD}^{M_n(x)[g]} = M_n(M_\infty, \Lambda),$$

where Λ is a partial iteration strategy for M_∞.

Open questions

Question

Is $\mathrm{HOD}^{L[x]}$ (without the generic G) a core model?

Proposition (Schlutzenberg, 2016)

Given sufficient large cardinals, there is a cone of reals x such that if $\mathcal{F}$ is a natural candidate for a limit system to analyze $\mathrm{HOD}^{L[x]}$, then $\mathcal{F}$ is not closed under pseudo-comparison of pairs.

Question

Is $\mathrm{HOD}^{M_n(x)}$ (without the generic g) a core model?

It is not even known if $\mathrm{HOD}^{L[x]}$ and $\mathrm{HOD}^{M_n(x)}$ are models of GCH.

Thank you for your attention!