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RAMSEY-LIKE CARDINALS

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Abstract. One of the numerous characterizations of a Ramsey cardinal κ involves the existence of certain types of elementary embeddings for transitive sets of size κ satisfying a large fragment of ZFC. I introduce new large cardinal axioms generalizing the Ramsey elementary embeddings characterization and show that they form a natural hierarchy between weakly compact cardinals and measurable cardinals. These new axioms serve to further our knowledge about the elementary embedding properties of smaller large cardinals, in particular those still consistent with $V = L$.

1. Introduction

Most large cardinals, including measurable cardinals and stronger notions, are defined in terms of the existence of elementary embeddings with that cardinal as the critical point. Several smaller large cardinals, such as weakly compact, indescribable, and Ramsey cardinals, have definitions in terms of elementary embeddings, but are more widely known for their other properties (combinatorial, reflecting, etc.). Other smaller large cardinals, such as ineffable and subtle cardinals, have no known elementary embedding characterizations. I will investigate the elementary embedding characterization of Ramsey cardinals and introduce new large cardinal axioms by generalizing the Ramsey embeddings. By placing these new large cardinals within the existing hierarchy, I try to shed light on the variety of elementary embedding properties that are possible for smaller large cardinals. In a forthcoming paper with Thomas Johnstone [GJ10], we use the new large cardinals to obtain some basic indestructibility results for Ramsey cardinals through the techniques of lifting embeddings. I hope that this project will motivate set theorists who work with smaller large cardinals to focus on investigating their elementary embedding properties.

Smaller large cardinals usually imply the existence of embeddings[1] for "mini-universes" of set theory of size κ and height above κ that I will call *weak κ-models* and *κ-models* of set theory. Let ZFC^- denote ZFC without the Powerset axiom. A transitive set $M \models \text{ZFC}^-$ of size κ with $\kappa \in M$ is a *weak κ-model* of set theory. A weak κ-model M is a *κ-model* if additionally $M^{<\kappa} \subseteq M$. Observe that for any cardinal κ, if $M \prec H_{\kappa^+}$ has size κ with $\kappa \subseteq M$, then M is a weak κ-model. Similarly, for regular $\lambda > \kappa$, if $X \prec H_\lambda$ has size κ with $\kappa+1 \subseteq X$, then the Mostowski collapse of X is a weak κ-model. So there are always many weak κ-models for any cardinal κ. If additionally $\kappa^{<\kappa} = \kappa$, we can use a Löwenheim-Skolem type construction to build κ-models $M \prec H_{\kappa^+}$ and substructures $X \prec H_\lambda$ whose collapse will be a κ-model.

[1]Throughout the paper, unless specifically stated otherwise, the sources and targets of elementary embeddings are assumed to be *transitive* sets or classes.

To provide a motivation for the new large cardinal notions I am introducing, I recall the various equivalent characterizations of weakly compact cardinals.[2]

Theorem 1.1. *If $\kappa^{<\kappa} = \kappa$, then the following are equivalent:*

1. (Compactness Property) *κ is weakly compact. That is, κ is uncountable and every $<\kappa$-satisfiable theory in a $L_{\kappa,\kappa}$ language of size at most κ is satisfiable.*
2. (Extension Property) *For every $A \subseteq \kappa$, there is a transitive structure W properly extending V_κ and $A^* \subseteq W$ such that $\langle V_\kappa, \in, A\rangle \prec \langle W, \in, A^*\rangle$.*
3. (Tree Property) *κ is inaccessible and every κ-tree has a cofinal branch.*
4. (Embedding Property) *Every $A \subseteq \kappa$ is contained in weak κ-model M for which there exists an elementary embedding $j : M \to N$ with critical point κ.*
5. *Every $A \subseteq \kappa$ is contained in a κ-model M for which there exists an elementary embedding $j : M \to N$ with critical point κ.*
6. *Every $A \subseteq \kappa$ is contained in a κ-model $M \prec H_{\kappa^+}$ for which there exists an elementary embedding $j : M \to N$ with critical point κ.*
7. *For every κ-model M, there exists an elementary embedding $j : M \to N$ with critical point κ.*
8. (Hauser Property) *For every κ-model M, there exists an elementary embedding $j : M \to N$ with critical point κ such that j and M are elements of N.*

Call an elementary embedding $j : M \to N$ between transitive models of ZFC$^-$ *κ-powerset preserving* if it has critical point κ and M and N have the *same* subsets of κ. Note that if $j : M \to N$ has critical point κ, then $\mathcal{P}(\kappa)^M \subseteq \mathcal{P}(\kappa)^N$, and so for such an embedding to be κ-powerset preserving, N must not acquire additional subsets of κ. For example, it is trivially true that any elementary embedding $j : V \to M$ with critical point κ is κ-powerset preserving since $M \subseteq V$. Many common set theoretic constructions involve building a directed system of elementary embeddings by iterating the ultrapower construction starting from a measure. The existence of the measure required for the second step of this iteration is equivalent to the ultrapower embedding by the initial measure being κ-powerset preserving. This makes κ-powerset preservation a necessary precondition for iterating the ultrapower construction (see Sections 2 and 4), and thus a natural notion to study. Another important motivation for focusing on κ-powerset preserving embeddings comes from a general trend in the theory of large cardinals of making the source and target of the embedding closely related to derive various reflection properties.

The general idea is to consider the elementary embedding characterizations of weakly compact cardinals from Theorem 1.1 (4)-(7) with the added assumption that the embeddings have to be *κ-powerset preserving*. We will soon see that this innocuous looking assumption destroys the equivalence in the strongest possible sense. We are now ready to introduce the new *Ramsey-like* large cardinal notions: *weakly Ramsey* cardinals, *strongly Ramsey* cardinals, *super Ramsey* cardinals, and *superlatively Ramsey* cardinals. These and related large cardinal notions will be the subject of this paper.

[2]The proofs of these equivalences are standard and parts of them can be found in [Kan03] (Ch 1, Sec. 4 and Ch. 2, Sec 7.) and [Cum10] (Sec. 16).

Definition 1.2. A cardinal κ is *weakly Ramsey* if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a κ-powerset preserving elementary embedding $j : M \to N$.

Recall that a cardinal κ is *Ramsey* if every coloring $f : [\kappa]^{<\omega} \to 2$ has a homogeneous set of size κ.[3] The connection between the new notions and Ramsey cardinals is seen in the following theorem implicit in [Dod82] and [Mit79].

Theorem 1.3. *A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a κ-powerset preserving elementary embedding $j : M \to N$ with the additional property that whenever $\langle A_n \mid n \in \omega\rangle$ are subsets of κ such that each $A_n \in M$ and $\kappa \in j(A_n)$, then $\cap_{n\in\omega} A_n \neq \emptyset$.*

For $\langle A_n \mid n \in \omega\rangle \in M$, the conclusion follows trivially by elementarity, so the content here is for sequences *not* in M. I will sketch a proof of Theorem 1.3 in Sections 3 and 4. We will see later that the measure derived from an embedding of Theorem 1.3 allows the ultrapower construction to be iterated through all the ordinals (see Section 5). Thus, there is an obvious gap between weakly Ramsey cardinals, where we can only begin iterating by constructing the second measure, and Ramsey cardinals, where we can already iterate the construction through all the ordinals. This gap will be filled by a refined hierarchy of new large cardinal notions which are introduced in Section 5 and form the subject of [GW10].

Definition 1.4. A cardinal κ is *strongly Ramsey* if every $A \subseteq \kappa$ is contained in a κ-model M for which there exists a κ-powerset preserving elementary embedding $j : M \to N$.

A motivation for introducing strongly Ramsey cardinals lies in the techniques used to demonstrate the indestructibility of large cardinals by certain kinds of forcing. Having $M^{<\kappa} \subseteq M$ makes it possible to use the standard techniques of lifting the embedding to a forcing extension. Note that strongly Ramsey cardinals are clearly Ramsey since every $\langle A_n \mid n \in \omega\rangle \subseteq M$ is an element of M.

Definition 1.5. A cardinal κ is *super Ramsey* if every $A \subseteq \kappa$ is contained in a κ-model $M \prec H_{\kappa^+}$ for which there exists a κ-powerset preserving elementary embedding $j : M \to N$.

A motivation for introducing super Ramsey cardinals is that having $M \prec H_{\kappa^+}$ guarantees that M is stationarily correct.

Definition 1.6. A cardinal κ is *superlatively Ramsey* if for every κ-model $M \prec H_{\kappa^+}$, there exists a κ-powerset preserving elementary embedding $j : M \to N$.

Since a weak κ-model can take the Mostowski collapse of any of its elements, Definitions 1.2, 1.4, 1.5, 1.6 and Theorem 1.3 hold not just for any $A \subseteq \kappa$, but more generally for any $A \in H_{\kappa^+}$.

The following theorem summarizes what is known about where the new large cardinals fit into the existing hierarchy. Also, see the diagram that follows.

Theorem 1.7.

(1) *A measurable cardinal is a super Ramsey limit of super Ramsey cardinals.*

[3] A set H is *homogeneous* for a coloring $f : [\kappa]^{<\omega} \to 2$ if H is homogeneous for $f \restriction [\kappa]^n$ for every $n \in \omega$. Note that the value of f on H can differ depending on n.

(2) *A super Ramsey cardinal is a strongly Ramsey limit of strongly Ramsey cardinals.*
(3) *A strongly Ramsey cardinal is a limit of completely Ramsey cardinals. It is Ramsey but not necessarily completely Ramsey.*[4]
(4) *A Ramsey cardinal is a weakly Ramsey limit of weakly Ramsey cardinals.*
(5) *Weakly Ramsey cardinals are consistent with $V = L$.*
(6) *A weakly Ramsey cardinal is a weakly ineffable limit of completely ineffable cardinals.*[5]
(7) *There are no superlatively Ramsey cardinals.*

The proofs of all statements excluding (4) and (5) are given in Section 3. Statements (4) and (5) are proved in the upcoming paper [GW10]. Theorem 1.7 shows, surprisingly, that the various embedding characterizations of weakly compact cardinals (Theorem 1.1) form a hierarchy of strength when the powerset preservation property is added. The most general such property is even inconsistent! This hierarchy fits quite naturally into the large cardinal hierarchy and suggests further refinements such as those introduced in Section 5.

The diagram on the next page illustrates how the new notions fit into the existing hierarchy. The solid arrows indicate direct implications and the dashed arrows indicate consistency strength.

[4] *Completely Ramsey* cardinals top a hierarchy of large cardinals generalizing Ramsey cardinals and were introduced in [Fen90].

[5] *Completely ineffable* cardinals were introduced in [Kle78].


Superlatively Ramsey
0=1
Strong
Measurable
Super Ramsey
Strongly Ramsey
Completely Ramsey
Ramsey
ω_1-Erdős
ω_1-iterable
$0^{\#}$ exists
$V = L$
$V = L$
α-iterable
Weakly Ramsey/1-iterable
Completely Ineffable
Ineffable
Weakly Ineffable
Subtle
Strongly Unfoldable
Unfoldable
Totally Indescribable
Weakly Compact

2. Preliminaries

The existence of an elementary embedding $j : V \to M$ with critical point κ is equivalent to the existence of a κ-complete ultrafilter[6] on κ. Similarly the existence of such an elementary embedding for a weak κ-model of set theory is equivalent to the existence of a certain filter measuring all the subsets of κ of the model. In this section, we will recall the various properties of such "mini" ultrafilters and use them to characterize the new Ramsey-like large cardinal notions.

Definition 2.1. Suppose M is a transitive model of ZFC$^-$ and κ is a cardinal in M. A set $U \subseteq \mathcal{P}(\kappa)^M$ is an M-*ultrafilter* if $\langle M, \in, U\rangle \models$ "U is a κ-complete normal ultrafilter".

Recall that an ultrafilter is κ-*complete* if the intersection of any $< \kappa$-sized collection of sets in the ultrafilter is itself an element of the ultrafilter. An ultrafilter is *normal* if every function regressive on a set in the ultrafilter is constant on a set in the ultrafilter, or equivalently if it is closed under diagonal intersections of length κ. By definition, M-ultrafilters are κ-complete and normal only from the *point of view* of M, that is, the collection of sets being intersected or diagonally intersected has to be an element of M. We will say that an M-ultrafilter is *countably complete* if every countable collection of sets in the ultrafilter has a nonempty intersection. Obviously, any M-ultrafilter is, by definition, countably complete from the point of view of M, but countable completeness requests the property to hold of *all* sequences, not just those in M.[7] Unless M satisfies some extra condition, such as being closed under countable sequences, an M-ultrafilter need not be countably complete. A modified version of the Łoś ultrapower construction using only functions on κ that are elements of M can be carried out with an M-ultrafilter. The ultrapower of V by an ultrafilter on κ is well-founded if and only if the ultrafilter is countably complete. For M-ultrafilters, there is no such nice characterization of well-founded ultrapowers. While countable completeness is sufficient, it is not necessary.[8]

Definition 2.2. Suppose M is a weak κ-model. An M-ultrafilter U on κ is 0-*good* if the ultrapower of M by U is well-founded.

The notion of 0-good M-ultrafilters anticipates the discussion in Section 5, where I introduce the generalized notion of α-*good* M-ultrafilters. A 0-good M-ultrafilter on κ gives rise to an elementary embedding of M with critical point κ, that is, the ultrapower embedding. The next proposition shows conversely that an elementary embedding of M with critical point κ gives rise to a 0-good M-ultrafilter on κ.

Proposition 2.3. *Suppose M is a weak κ-model and $j : M \to N$ is an elementary embedding with critical point κ. If we let $X = \{j(f)(\kappa) \mid\ f : \kappa \to M \text{ and } f \in M\}$ and $\pi : X \to K$ be the Mostowski collapse, then $h = \pi \circ j : M \to K$ is an elementary embedding with critical point κ having the following properties:*

(1) *h is the ultrapower by an M-ultrafilter on κ,*

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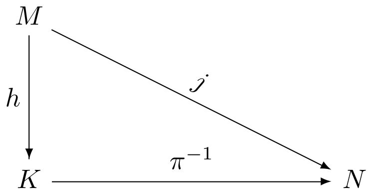
[6]Throughout the paper, filters are assumed to be nonprincipal.

[7]It is more standard for countable completeness to mean ω_1-completeness which requires the intersection to be an element of the ultrafilter. However, the weaker notion we use here is better suited to M-ultrafilters because the countable collection itself can be external to M, and so there is no reason to suppose the intersection to be an element of M.

[8]See [Kan03] (Ch. 4, Sec. 19) for details on ultrapowers by M-ultrafilters.

(2) *the target K has size κ,*
(3) *we get the commutative diagram:*



(4) *if j was κ-powerset preserving, then so is h,*
(5) *if $M^\alpha \subseteq M$ for some ordinal α, then $K^\alpha \subseteq K$,*
(6) *$\kappa \in j(A)$ if and only if $\kappa \in h(A)$ for all $A \in \mathcal{P}(\kappa)^M$.*

Proof. Properties (2) and (3) follow directly from the definition of h. Define $U = \{A \in \mathcal{P}(\kappa)^M \mid \kappa \in j(A)\}$. It follows by standard arguments that U is an M-ultrafilter, the embedding h is the ultrapower by U, and if $M^\alpha \subseteq M$, then $K^\alpha \subseteq K$. This gives properties (1) and (5). For (4), observe that $\pi(\beta) = \beta$ for all $\beta \leq \kappa$, and hence the critical point of π^{-1} is above κ. This implies that N and K have the same subsets of κ. Finally, for (6), we have

$$\kappa \in j(A) \leftrightarrow \pi(\kappa) \in \pi \circ j(A) \leftrightarrow \kappa \in \pi \circ j(A) \leftrightarrow \kappa \in h(A).$$

□

By Proposition 2.3, we can assume, without loss of generality, in Definitions 1.2, 1.4, 1.5, 1.6, and Theorem 1.3 that $j : M \to N$ is the *ultrapower* by an M-ultrafilter on κ, the target N has size κ, and if $M^\alpha \subseteq M$ for some ordinal α, then $N^\alpha \subseteq N$.

Definition 2.4. Suppose M is a weak κ-model. An M-ultrafilter U on κ is *weakly amenable* if for every $A \in M$ of size κ in M, the intersection $U \cap A$ is an element of M.

Equivalently, U is weakly amenable if for every sequence $\langle B_\alpha \mid \alpha < \kappa\rangle$ in M of subsets of κ, the set $\{\alpha < \kappa \mid B_\alpha \in U\}$ is an element of M.

It follows directly from the definition that if U is weakly amenable, then for every $A \subseteq \kappa^n \times \kappa$ in M, the set $\{\overrightarrow{\alpha} \in \kappa^n \mid A_{\overrightarrow{\alpha}} \in U\} \in M$.[9] This allows us to define *product* ultrafilters U^n on $\mathcal{P}(\kappa^n)^M$ for every $n \in \omega$. We define U^n by induction on n by $A \subseteq \kappa^n \times \kappa$ is in $U^{n+1} = U^n \times U$ if $A \in M$ and $\{\overrightarrow{\alpha} \in \kappa^n \mid A_{\overrightarrow{\alpha}} \in U\} \in U^n$. Note that weak amenability is clearly a prerequisite for defining product ultrafilters. The next proposition is a standard fact about product ultrafilters that will be used later.

Proposition 2.5. *Suppose M is a weak κ-model and U is a weakly amenable M-ultrafilter on κ. Then for every $A \in U^n$, there is $B \in U$ such that if $\alpha_1 < \ldots < \alpha_n$ are in B, then $\langle \alpha_1, \ldots, \alpha_n\rangle \in A$.*

Proof. I will argue by induction on $n \in \omega$. For the base case $n = 2$, fix $A \in U^2$. By definition of U^2, the set $X = \{\alpha < \kappa \mid A_\alpha \in U\} \in U$. In M, define a sequence $\langle C_\alpha \mid \alpha < \kappa\rangle$ of sets in U by $C_\alpha = A_\alpha$ for $\alpha \in X$ and $C_\alpha = \kappa$ otherwise. The diagonal intersection $D = \triangle_{\alpha<\kappa} C_\alpha$ is in U by normality. I claim that $B = D \cap X$ has the requisite property for A. If $\alpha_1 < \alpha_2 \in B$, then $\alpha_1 \in X$, and so $\alpha_2 \in C_{\alpha_1} = A_{\alpha_1}$. It follows that $\langle \alpha_1, \alpha_2\rangle \in A$.

[9]Whenever $A \subseteq X \times Y$, we will use the notation A_a to denote the set $\{y \in Y \mid (a, y) \in A\}$.

Now suppose inductively that the statement holds for U^n, and fix $A \in U^{n+1}$. By definition of U^{n+1}, the set $X = \{\vec{\alpha} \in \kappa^n \mid A_{\vec{\alpha}} \in U\} \in U^n$. By the inductive assumption, fix $Y \in U$ such that for all $\alpha_1 < \cdots < \alpha_n$ in Y, the sequence $\langle \alpha_1, \ldots, \alpha_n \rangle \in X$. It follows that for all $\alpha_1 < \cdots < \alpha_n$ in Y, the set $A_{\langle \alpha_1, \ldots, \alpha_n \rangle} \in U$. In M, we define a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ of sets in U as follows. For $\alpha \in Y$, let Seq^α be the collection of all increasing n-tuples of ordinals in Y ending in α. Define $C_\alpha = \cap_{\vec{\beta} \in Seq^\alpha} A_{\vec{\beta}}$ for $\alpha \in Y$ and $C_\alpha = \kappa$ otherwise. The diagonal intersection $D = \triangle_{\alpha < \kappa} C_\alpha$ is in U by normality. I claim that $B = D \cap Y$ has the requisite property for A. If $\alpha_1 < \cdots < \alpha_n < \alpha_{n+1} \in B$, then $\alpha_1 < \cdots < \alpha_n \in Y$ and $\alpha_{n+1} \in C_{\alpha_n}$. Since $\alpha_n \in Y$, we have $C_{\alpha_n} = \cap_{\vec{\beta} \in Seq^{\alpha_n}} A_{\vec{\beta}}$. So in particular, $\alpha_{n+1} \in A_{\langle \alpha_1, \ldots, \alpha_n \rangle}$, and hence $\langle \alpha_1, \ldots, \alpha_n, \alpha_{n+1} \rangle \in A$. This completes the induction step and finishes the argument. □

The next proposition reformulates the property of κ-powerset preservation in terms of the existence of weakly amenable M-ultrafilters.

Proposition 2.6. *Suppose M is a transitive model of* ZFC$^-$.

1. *If $j : M \to N$ is the ultrapower by a weakly amenable M-ultrafilter on κ, then j is κ-powerset preserving.*
2. *If $j : M \to N$ is a κ-powerset preserving embedding, then the M-ultrafilter $U = \{A \in \mathcal{P}(\kappa)^M \mid \kappa \in j(A)\}$ is weakly amenable.*

See [Kan03] (Ch. 4, Sec. 19) for proof.

Definition 2.7. Suppose M is a weak κ-model. An M-ultrafilter on κ is 1-*good* if it is 0-good and weakly amenable.

Now we are ready to characterize the Ramsey-like large cardinal notions in terms of the existence of M-ultrafilters.

Proposition 2.8.

1. *Suppose $\kappa^{<\kappa} = \kappa$, then κ is weakly compact if and only if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a* 0*-good M-ultrafilter on κ.*
2. *A cardinal κ is weakly Ramsey if and only if every $A \subseteq \kappa$ is contained a weak κ-model M for which there exists a* 1*-good M-ultrafilter on κ.*
3. *A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a weakly amenable countably complete M-ultrafilter on κ.*
4. *A cardinal κ is strongly Ramsey if and only if every $A \subseteq \kappa$ is contained in a κ-model M for which there exists a weakly amenable M-ultrafilter on κ.*
5. *A cardinal κ is super Ramsey if and only if every $A \subseteq \kappa$ is contained in a κ-model $M \prec H_{\kappa^+}$ for which there exists a weakly amenable M-ultrafilter on κ.*

Proof. Statement (1) follows from Theorem 1.1(4) and Proposition 2.3. Statement (2) follows from Proposition 2.3 and Proposition 2.6. Statement (3) follows Proposition 2.3 and the fact that countably complete M-ultrafilters are 0-good. Statements (4) and (5) follow because for a κ-model M, an M-ultrafilter is always countably complete. □

We will say that a weak κ-model $M \models$"I am H_{κ^+}" if M thinks all its sets have transitive closure of size at most κ. By replacing M with $\bar{M} = \{a \in M : M \models$ transitive closure of a has size $\leq \kappa\}$, we can assume, without loss of generality, that $M \models$"I am H_{κ^+}" in every statement of Proposition 2.8. The model $\bar{M}$ is a weak κ-model that will be as closed under $< \kappa$-sequences as M and the M-ultrafilter U will retain the necessary properties when viewed as an $\bar{M}$-ultrafilter.

3. The Large Cardinal Hierarchy

In this section, I prove parts (1)-(3) and (6)-(7) of Theorem 1.7. I also show the backward direction of Theorem 1.3 as restated in Proposition 2.8(3). The arguments below rely mainly on the powerful reflecting properties of κ-powerset preserving embeddings. It is also important to note that if two transitive models of ZFC$^-$ have the same subsets of κ, then they also have the same sets with transitive closure of size at most κ.

Proposition 3.1. *Weakly Ramsey cardinals are inaccessible.*

Proof. Suppose κ is weakly Ramsey. If κ is not regular, then there is a cofinal map $f : \alpha \to \kappa$ for some $\alpha < \kappa$. Choose a weak κ-model M containing f for which there is an embedding $j : M \to N$ with critical point κ. Now observe that $j(f) = f$ and $j(f)$ must be cofinal in $j(\kappa)$ by elementarity, which is a contradiction. If κ is not a strong limit, then there is $\alpha < \kappa$ with $|\mathcal{P}(\alpha)| \geq \kappa$. Fix $f : \kappa \overset{1\text{-}1}{\to} \mathcal{P}(\alpha)$ and choose a weak κ-model M containing f for which there is a κ-powerset preserving embedding $j : M \to N$. Let $j(f)(\kappa) = A \subseteq \alpha$. The set $A \in M$ by κ-powerset preservation, and so $j(f)(\kappa) = A = j(A)$ since $A \subseteq \alpha$. Thus, $N \models \exists \xi < j(\kappa)\, j(f)(\xi) = j(A)$, and so by elementarity, $M \models \exists \xi < \kappa\, f(\xi) = A$. Now we have $j(f)(\xi) = f(\xi) = A$ and $j(f)(\kappa) = A$, which contradicts that $j(f)$ is 1-1 by elementarity. $\square$

Notice that κ-powerset preservation is not required to show regularity. However, without κ-powerset preservation it would be impossible to show that κ is a strong limit since property (4) of weakly compact cardinals from Theorem 1.1 without $\kappa^{<\kappa} = \kappa$ does not imply that κ is a strong limit. To show this we start with a weakly compact cardinal κ and force to add κ^+ many Cohen reals. In the forcing extension, κ is clearly no longer a strong limit but it can be shown that it retains property (4) from Theorem 1.1.[10]

Definition 3.2. An uncountable regular cardinal κ is *weakly ineffable* if for every sequence $\langle A_\alpha \mid \alpha \in \kappa \rangle$ with $A_\alpha \subseteq \alpha$, there exists $A \subseteq \kappa$ such that the set $S = \{\alpha \in \kappa \mid A \cap \alpha = A_\alpha\}$ has size κ. An uncountable regular cardinal κ is *ineffable* if such A can be found for which the corresponding set S is stationary.

Ineffable cardinals are limits of weakly ineffable cardinals. This follows since ineffable cardinals are Π^1_2-indescribable and being weakly ineffable is a Π^1_2- statement satisfied by ineffable cardinals. Ramsey cardinals are limits of ineffable cardinals but need not be ineffable. Since being Ramsey is a Π^1_2-statement, a Ramsey cardinal that is ineffable is a limit of Ramsey cardinals. In particular, the least Ramsey cardinal cannot be ineffable. Like Ramsey cardinals, ineffable cardinals can be characterized by the existence of homogeneous sets for colorings. An uncountable

[10]The argument is due to Hamkins and will appear in [Ham07] (Ch. 6).

cardinal κ is ineffable if and only if every coloring $f : [\kappa]^2 \to 2$ is homogeneous on a stationary set.[11]

Theorem 3.3. *Weakly Ramsey cardinals are weakly ineffable.*

Proof. Suppose κ is weakly Ramsey. Fix $\vec{A} = \langle A_\alpha \mid \alpha \in \kappa\rangle$ with each $A_\alpha \subseteq \alpha$. Choose a weak κ-model M containing $\vec{A}$ for which there exists a κ-powerset preserving embedding $j : M \to N$. Consider $j(\vec{A})$ and, in particular, $A = j(\vec{A})(\kappa)$. Since $A \subseteq \kappa$, by κ-powerset preservation, $A \in M$. It is easy to see that the set $S = \{\alpha \in \kappa \mid A \cap \alpha = A_\alpha\}$ is stationary in M. Fix a club $C \in M$ and observe that $\kappa \in j(S) \cap j(C)$. In particular, S has size κ. So we have shown that κ is weakly ineffable. □

Since Ramsey cardinals are weakly Ramsey, it follows that not every weakly Ramsey cardinal is ineffable. However, consistency strength-wise, weakly Ramsey cardinals are stronger than ineffable cardinals. I show below that weakly Ramsey cardinals are limits of completely ineffable cardinals, where complete ineffability is a strengthening of ineffability introduced by [Kle78].

Definition 3.4. A collection $R \subseteq \mathcal{P}(\kappa)$ is a *stationary class* if

1. $R \neq \emptyset$,
2. for all $A \in R$, A is stationary in κ,
3. if $A \in R$ and $B \supseteq A$, then $B \in R$.

Definition 3.5. A cardinal κ is *completely ineffable* if there is a stationary class R such that for every $A \in R$ and $f : [A]^2 \to 2$, there is $H \in R$ homogeneous for f.

In particular, since $\kappa \in R$ for every stationary class R, it follows that if κ is completely ineffable, then every $f : [\kappa]^2 \to 2$ has a homogeneous set that is stationary in κ. Thus, completely ineffable cardinals are clearly ineffable.

Lemma 3.6. *If M is a weak κ-model, U is a 1-good M-ultrafilter on κ, $A \in U$, and $f : [A]^{<\omega} \to 2$ is in M, then for every $n \in \omega$, there is $H_n \in U$ homogeneous for $f \restriction [A]^n$.*

Proof. Recall that 1-good ultrafilters are weakly amenable, and so we can define the product ultrafilters U^n for every $n \in \omega$. It is easy to see that either $\{\vec{\alpha} \in [A]^n \mid f(\vec{\alpha}) = 0\} \in U^n$ or $\{\vec{\alpha} \in [A]^n \mid f(\vec{\alpha}) = 1\} \in U^n$. Call X the one of the above sets which is in U^n. By Proposition 2.5, there is $B \in U$ such that for all $\alpha_1 < \cdots < \alpha_n$ in B, the sequence $\langle \alpha_1, \ldots, \alpha_n\rangle \in X$. Clearly, letting $B = H_n$ works. □

Theorem 3.7. *Weakly Ramsey cardinals are limits of completely ineffable cardinals.*

Proof. Suppose κ is weakly Ramsey. Using Proposition 2.8(2), fix a weak κ-model M containing V_κ for which there exists a 1-good M-ultrafilter U on κ, and let $j : M \to N$ be the κ-powerset preserving ultrapower by U. I will argue that $N \models$ "κ is completely ineffable". Thus, for every $\alpha < \kappa$, the model N will satisfy that there is a completely ineffable cardinal between α and $j(\kappa)$. By elementarity, M will satisfy that there is a completely ineffable cardinal between α and κ. Since M contains V_κ, it will be correct about this.

[11] For an introduction to ineffability see [Dev84] (Ch. 7, Sec. 2).

To show that κ is completely ineffable in N, we need to build in N a stationary class R satisfying the property of Definition 3.5. Working inside N, we will define a sequence of sets $\langle R_\alpha \mid \alpha \in Ord^N\rangle$ as follows. Define R_0 to be the collection of all stationary subsets of κ. Inductively, given R_α, define $R_{\alpha+1}$ to be the set of all $A \in R_\alpha$ such that for every $f : [A]^2 \to 2$, there is $H \in R_\alpha$ homogeneous for f. At limits take intersections. Since the R_α form a decreasing sequence, there is θ such that $R_\theta = R_{\theta+1}$. Letting $R = R_\theta$, we argue that it is a stationary class. Note that R satisfies the property of Definition 3.5 by construction. It is clear that R consists of stationary sets and is closed under supersets. We verify that R is nonempty by showing that $U \subseteq R$. Since all elements of U are stationary in N, it follows that $U \subseteq R_0$. Inductively suppose $U \subseteq R_\alpha$. Fix $A \in U$ and $f : [A]^2 \to 2$ in N. By κ-powerset preservation, $f \in M$, and so by Lemma 3.6 there is $H \in U$ homogeneous for f making $A \in R_{\alpha+1}$. This completes the argument that $U \subseteq R$, and hence $R \neq \emptyset$. □

Theorems 3.3 and 3.7 establish Theorem 1.7(6).

Proposition 3.8. *If κ is a weakly Ramsey cardinal, then $\diamondsuit_\kappa$ holds.*

Proof. A weakly Ramsey cardinal is weakly ineffable and $\diamondsuit_\kappa$ holds for weakly ineffable cardinals.[12] □

Theorem 3.9. *Ramsey cardinals are limits of weakly Ramsey cardinals.*

The theorem is included here for completeness of presentation. The proof relies on the properties of the refined hierarchy of Ramsey-like large cardinal notions introduced in Section 5. The hierarchy starts with weakly Ramsey cardinals, has length $\omega_1 + 1$ and is bounded above by Ramsey cardinals. In [GW10], we show that every large cardinal in the hierarchy is a limit of each of the large cardinals in the hierarchy below it. The theorem follows immediately from this fact.

The next theorem establishes the backward direction of Proposition 2.8(3), and hence of Theorem 1.3. The forward direction is a much more complicated argument and will be discussed in detail in Section 4.

Theorem 3.10. *If every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a weakly amenable countably complete M-ultrafilter on κ, then κ is Ramsey.*

Proof. Fix $f : [\kappa]^{<\omega} \to 2$ and put it into a weak κ-model M for which there exists a weakly amenable countably complete M-ultrafilter U on κ. Recall that, by definition of countable completeness, every countable sequence of sets in U has a nonempty intersection. In fact, the intersection must have size κ. To see this, suppose to the contrary that A_n for $n \in \omega$ are elements of U and $A = \cap_{n\in\omega} A_n$ is bounded by some $\alpha < \kappa$. Since $\kappa \setminus \alpha$ is an element of U, the intersection of all A_n together with $\kappa \setminus \alpha$ must be nonempty by countable completeness of U, but this is obviously false. Thus, indeed, the intersection must have size κ. By Lemma 3.6, for every $n \in \omega$, there is $H_n \in U$ homogeneous for $f \restriction [\kappa]^n$. The intersection $H = \cap_{n\in\omega} H_n$ is a set of size κ homogeneous for f. □

Theorem 3.11. *Strongly Ramsey cardinals are limits of Ramsey cardinals.*

[12] See [Dev84] (Ch. 7, Sec. 2) for a proof that $\diamondsuit_\kappa$ holds for ineffable κ and observe that weak ineffability suffices.

Proof. Suppose κ is strongly Ramsey. Using Proposition 2.8, fix a κ-model M for which there is a weakly amenable M-ultrafilter U on κ and let $j : M \to N$ be the κ-powerset preserving ultrapower by U. Recall that since M is a κ-model, U must be countably complete. Also, κ-models always contain V_κ as an element. I will argue that $N \models$ "κ is Ramsey". As before, this suffices to show that κ is a limit of Ramsey cardinals. Fix $f : [\kappa]^{<\omega} \to 2$ in N and observe that it is in M by κ-powerset preservation. By Lemma 3.6, for every $n \in \omega$, there is $H_n \in U$ homogeneous for $f \restriction [\kappa]^n$. Since $M^{<\kappa} \subseteq M$, the sequence $\langle H_n : n \in \omega\rangle$ is an element of M, and hence $H = \cap_{n\in\omega} H_n$ is an element of M. Clearly H is an element of N as well. □

Note that for the proof above it clearly suffices that $M^\omega \subseteq M$. Thus, if every $A \subseteq \kappa$ is contained in a weak κ-model M with $M^\omega \subseteq M$ for which there exists a κ-powerset preserving embedding, then κ is a limit of Ramsey cardinals. It is an interesting open question whether it suffices to assume that $\mathrm{cf}^V(\kappa^+)^M \geq \omega_1$ (if κ is the largest cardinal in M, then $(\kappa^+)^M = Ord^M$).

Question 3.12. Suppose every $A \subseteq \kappa$ is contained in a weak κ-model M with $\mathrm{cf}^V(\kappa^+)^M \geq \omega_1$ for which there exists a κ-powerset preserving embedding $j : M \to N$. Does the existence of such a cardinal κ carry more strength than a Ramsey cardinal?

If $2^\kappa = \kappa^+$ in N, then the answer is affirmative. Under this assumption, I can show that κ is Ramsey in N. First, by Proposition 2.3, we can assume that $j : M \to N$ is the ultrapower by a 1-good M-ultrafilter U on κ without losing $2^\kappa = \kappa^+$ in the target model. Fixing $f : [\kappa]^{<\omega} \to 2$ in N, we know as before that for every $n \in \omega$, there is a set H_n in U homogeneous for $f \restriction [\kappa]^n$. Since $2^\kappa = \kappa^+$ holds in N, we can define in N an elementary chain of length κ^+ of transitive models of size κ: $X_0 \prec X_1 \prec \cdots \prec X_\alpha \prec \cdots \prec H^N_{\kappa^+}$ whose union is $H^N_{\kappa^+}$. By assumption $(\kappa^+)^N = (\kappa^+)^M$ has uncountable cofinality, and so all H_n must be contained in some X_α. Since each $X_\xi \in M$, the weak amenability of U implies that $u = U \cap X_\alpha$ is an element of M. Since all H_n are contained in u, the model M satisfies that for every $n \in \omega$, u contains a set homogeneous for $f \restriction [\kappa]^n$. It follows that there is a sequence $\langle H'_n \mid n \in \omega\rangle$ of such sets in u that is an element of M. The intersection $H = \cap_{n\in\omega} H'_n$ is an element of U since $u \subseteq U$ and U is κ-complete for sequences in M by the definition of M-ultrafilter. Thus, in particular, H has size κ.

Next, I show that strongly Ramsey cardinals are limits of the *completely Ramsey* cardinals that top Feng's Π_α-Ramsey hierarchy [Fen90]. If I is an ideal containing all the non-stationary subsets of κ, let $\mathscr{R}^+(I)$ be the collection of all $X \subseteq \kappa$ such that every $f : [X]^{<\omega} \to 2$ has a homogeneous set in $\mathcal{P}(\kappa) \setminus I$. Define an operation $\mathscr{R}$ on such ideals by $\mathscr{R}(I) = \mathcal{P}(\kappa) \setminus \mathscr{R}^+(I)$. Feng showed that the $\mathscr{R}$ operation applied to an ideal always yields an ideal and iterated it to define a hierarchy of ideals on κ as follows. Let I_0 be the ideal of non-stationary subsets of κ. Define $I_{\alpha+1} = \mathscr{R}(I_\alpha)$ and $I_\lambda = \cup_{\alpha<\lambda} I_\alpha$.[13] A cardinal κ is *completely Ramsey* if for all α, we have $\kappa \notin I_\alpha$. Notice the similarity here to the completely ineffable cardinals and the fact that it easily follows that completely Ramsey cardinals are completely ineffable. Also, completely Ramsey cardinals are clearly Ramsey since $\kappa \notin I_1$ implies that every $f : [\kappa]^{<\omega} \to 2$ has a homogeneous set that is stationary in κ.

[13]Feng's definition of I_n for $n \in \omega$ is slightly more subtle, but for our purposes here this simplified version suffices.

Theorem 3.13. *Strongly Ramsey cardinals are limits of completely Ramsey cardinals.*

Proof. Suppose κ is strongly Ramsey. By Proposition 2.8, fix a κ-model M for which there is a weakly amenable M-ultrafilter U on κ, and let $j : M \to N$ be the κ-powerset preserving ultrapower by U. Now argue that U is contained in the complement of every I_α as defined in N. Thus, $N \models$ "$\kappa \notin I_\alpha$ for all α", and so κ is completely Ramsey in N. As usual, this suffices. □

In [GJ10], we show that it is consistent that there is a strongly Ramsey cardinal that is not ineffable. This follows by showing that it is consistent to have a strongly Ramsey κ with a slim κ-Kurepa tree. Ineffable cardinals can never have slim Kurepa trees. Since completely Ramsey cardinals are in particular ineffable, this implies that a strongly Ramsey cardinal need not be completely Ramsey.

Theorem 3.13 together with the remarks above establish Theorem 1.7(3).

If κ is strongly Ramsey but not ineffable, it will not be possible to put every $A \subseteq \kappa$ into a stationarily correct κ-model for which there exists a κ-powerset preserving embedding. A κ-model for which there exists an κ-powerset preserving embedding always believes that κ is ineffable and if it is stationarily correct, it will be correct about this. This motivates the definition of super Ramsey cardinals which guarantee that we can always get a κ-model that is stationarily correct. In particular, note that super Ramsey cardinals are ineffable.

Theorem 3.14. *Super Ramsey cardinals are limits of strongly Ramsey cardinals.*

Proof. Suppose κ is super Ramsey. Choose a κ-model $M \prec H_{\kappa^+}$ for which there exists a κ-powerset preserving embedding $j : M \to N$. As usual, it suffices to show that $N \models$ "κ is strongly Ramsey". By Proposition 2.3, we can assume, without loss of generality, that the strong Ramsey embeddings have targets of size κ. It follows that $H_{\kappa^+} \models$ "κ is strongly Ramsey", and therefore $M \models$ "κ is strongly Ramsey" by elementarity. By κ-powerset preservation, N has no new subsets of κ, and so it must agree that κ is strongly Ramsey. □

Theorem 3.14 establishes Theorem 1.7(2).

Theorem 3.15. *There are no superlatively Ramsey cardinals.*

Proof. Suppose that there exists a superlatively Ramsey cardinal and let κ be the *least* superlatively Ramsey cardinal. Choose any κ-model $M \prec H_{\kappa^+}$ and a κ-powerset preserving embedding $j : M \to N$. The strategy will be to show that κ is superlatively Ramsey in N. Observe first that $H^N_{\kappa^+} = M$. Thus, to show that κ is superlatively Ramsey in N, we need to verify in N that every κ-model $m \prec M$ has a κ-powerset preserving embedding. So let $m \in N$ be a κ-model such that $m \prec M$. Observe that $m \in M$ and $m \prec H_{\kappa^+}$. By Proposition 2.3, H_{κ^+} contains a κ-powerset preserving embedding for m. By elementarity, M contains some κ-powerset preserving embedding $h : m \to n$. Clearly $h \in N$ as well. Thus, κ is superlatively Ramsey in N. It follows that there is a superlatively Ramsey cardinal α below κ. This is, of course, impossible since we assumed that κ was the least superlatively Ramsey cardinal. □

Theorem 3.15 establishes Theorem 1.7(7).

Theorem 3.15 is surprising since, as was pointed out earlier, the embedding properties described in Definitions 1.2, 1.4, 1.5, and 1.6 without κ-powerset preservation are equivalent modulo the assumption that $\kappa^{<\kappa} = \kappa$. Once we add the powerset condition on the embeddings, the equivalence is strongly violated. Of course, now the question arises whether there can be any super Ramsey cardinals.

Theorem 3.16. *If κ is a measurable cardinal, then κ is a super Ramsey limit of super Ramsey cardinals.*

Proof. Suppose κ is a measurable cardinal. Fix $A \subseteq \kappa$ and a normal κ-complete ultrafilter U on κ. Take the structure $\langle H_{\kappa^+}, \in, U\rangle$ and using a Löwenheim-Skolem type construction, find an elementary substructure $\langle N, \in, U \cap N\rangle$ such that N is a κ-model and $A \in N$. Since U is a weakly amenable H_{κ^+}-ultrafilter, it follows by elementarity that $U \cap N$ is a weakly amenable N-ultrafilter. Thus, we found a κ-model N containing A for which there exists a weakly amenable N-ultrafilter on κ. By Proposition 2.8, this completes the proof that measurable cardinals are super Ramsey. To see that they are limits of super Ramsey cardinals, observe that if $j : V \to M$ is an elementary embedding with critical point κ, then both N and $N \cap U$ from the construction above are elements of M by the virtue of having transitive closure of size κ. So M agrees that κ is super Ramsey, and it follows that κ is a limit of super Ramsey cardinals in V. □

Theorem 3.16 completes the proof of Theorem 1.7(1).

Corollary 3.17. *Con*(ZFC + $\exists$ *measurable cardinal*) $\Longrightarrow$ *Con*(ZFC + $\exists$ *proper class of super Ramsey cardinals*)

It is an interesting observation that the Ramsey-like cardinals are incompatible with the Hauser property of weakly compact cardinals (Theorem 1.1(8)). The Hauser property plays a key role in many indestructibility by forcing arguments.

Proposition 3.18. *If a cardinal κ has the property that every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a κ-powerset preserving embedding $j : M \to N$ such that $j``\mathcal{P}(\kappa)^M$ is an element of N, then κ is a limit of measurable cardinals.*

Proof. Fix a weak κ-model M containing V_κ for which there exists a κ-powerset preserving embedding $j : M \to N$ such that $X = j``\mathcal{P}(\kappa)^M$ is an element of N. Consider the usual M-ultrafilter $U = \{B \in \mathcal{P}(\kappa)^M \mid \kappa \in j(B)\}$. Since $X \in N$, we can define the set $\{C \cap \kappa \mid C \in X \text{ and } \kappa \in C\} = \{B \subseteq \kappa \mid \kappa \in j(B)\} = U$ in N. Therefore U is an element of N. By the κ-powerset preservation property, U is also an N-ultrafilter, and hence N thinks that κ is measurable. It follows that κ must be a limit of measurable cardinals. □

Thus, having j as an element of N pushes the large cardinal strength beyond a measurable cardinal. For instance, if κ is 2^κ-supercompact, then κ will have the above property. To see this, fix a 2^κ-supercompact embedding $j : V \to M$ and $A \subseteq \kappa$. Choose some cardinal λ such that $j(\lambda) = \lambda$ and $j``2^\kappa \in H^M_{\lambda^+}$. We first restrict j to a set embedding $j : H_{\lambda^+} \to H^M_{\lambda^+}$ and observe that $H^M_{\lambda^+} \subseteq H_{\lambda^+}$. Thus, it makes sense to consider the structure $\langle H_{\lambda^+}, H^M_{\lambda^+}, j\rangle$. Take an elementary substructure $\langle K', N', h'\rangle$ of $\langle H_{\lambda^+}, H^M_{\lambda^+}, j\rangle$ of size κ such that $A \in K'$, $j``2^\kappa \in K'$, $\kappa+1 \subseteq K'$, and $K'^{<\kappa} \subseteq K'$. Let $\pi : K' \to K$ be the Mostowski collapse, $\pi``N' = N$, and $h = \pi``h'$. Observe that $N$ is the Mostowski collapse of $N'$. It is easy to check that $K$ is a $\kappa$-model containing $A$, the map $h : K \to N$ is a $\kappa$-powerset preserving embedding, and $h``\mathcal{P}(\kappa)^K$ is an element of N.

Recall from Theorem 1.1(2) that weakly compact cardinals can be characterized by the *extension property.* Finally, I will show that weakly Ramsey cardinals also have an extension-like property. Suppose $\mathfrak{X} \subseteq \mathcal{P}(\kappa)$. The structure $\langle V_\kappa, \in, B\rangle_{B\in\mathfrak{X}}$ will be the structure in the language consisting of $\in$ and unary predicate symbols for every element of $\mathfrak{X}$ with the natural interpretation.

Theorem 3.19. *A cardinal κ is weakly Ramsey if and only if every $A \subseteq \kappa$ belongs to a collection $\mathfrak{X} \subseteq \mathcal{P}(\kappa)$ such that the structure $\langle V_\kappa, \in, B\rangle_{B\in\mathfrak{X}}$ has a proper transitive elementary extension $\langle W, \in, B^*\rangle_{B\in\mathfrak{X}}$ with $\mathcal{P}(\kappa)^W = \mathfrak{X}$.*

Proof. ($\Longrightarrow$): Suppose that κ is weakly Ramsey and $A \subseteq \kappa$. Fix a weak κ-model M containing A and V_κ for which there exists a κ-powerset preserving embedding $j : M \to N$. If we let $\mathfrak{X} = \mathcal{P}(\kappa)^M$, then $\langle V_\kappa, \in, B\rangle_{B\in\mathfrak{X}} \prec \langle V_{j(\kappa)}, \in, j(B)\rangle_{B\in\mathfrak{X}}$.
($\Longleftarrow$): Fix $A \subseteq \kappa$. The set A belongs to a collection $\mathfrak{X} \subseteq \mathcal{P}(\kappa)$ such that the structure $\langle V_\kappa, \in, B\rangle_{B\in\mathfrak{X}}$ has a proper transitive elementary extension $\langle W, \in, B^*\rangle_{B\in\mathfrak{X}}$ with $\mathcal{P}(\kappa)^W = \mathfrak{X}$. We can assume that W has size κ since if this is not the case, we can take an elementary substructure of size κ which contains V_κ as a subset and collapse it. Since V_κ satisfies that H_{α^+} exists for every $\alpha < \kappa$, it follows by elementarity that H_{κ^+} exists in W.

Let $M = H^W_{\kappa^+}$ and observe that M is a weak κ-model containing A. Define $U = \{B \in \mathfrak{X} \mid \kappa \in B^*\}$. I claim that U is a 1-good M-ultrafilter. By Proposition 2.8, we will be done if we can verify the claim. Thus, we need to verify that U is a weakly amenable M-ultrafilter and that the ultrapower by U is well-founded.

It is clear that $\langle M, \in, U\rangle \models$ "U is an ultrafilter". To check that $\langle M, \in, U\rangle \models$ "U is normal", fix a regressive $f : B \to \kappa$ in M for some $B \in U$. Since we can code f as a subset of κ and $\mathcal{P}(\kappa)^W = \mathfrak{X}$, we can think of f as being in $\mathfrak{X}$. Now we can consider the regressive $f^* : B^* \to \kappa^*$ and let $f^*(\kappa) = \alpha < \kappa$ since $\kappa \in B^*$. Thus, $\kappa \in C^*$ where $C = \{\xi \in \kappa \mid f(\xi) = \alpha\}$, and hence $C \in U$. Thus $\langle M, \in, U\rangle \models$ "U is normal". It is a standard exercise to show that a normal ultrafilter on κ containing all the tail subsets of κ is κ-complete, and therefore since this property easily holds of U, we have $\langle M, \in, U\rangle \models$ "U is κ-complete". This completes the argument that U is an M-ultrafilter.

To show that U is weakly amenable, fix $\langle B_\alpha \mid \alpha < \kappa\rangle$ a sequence in M of subsets of κ. We need to see that the set $C = \{\alpha \in \kappa \mid B_\alpha \in U\}$ is in M. Again, since we can code the sequence $\langle B_\alpha \mid \alpha < \kappa\rangle$ as a subset of κ, we think of it as being in $\mathfrak{X}$. Thus, in W, we can define the set $\{\alpha \in \kappa \mid \kappa \in B^*_\alpha\}$, and it is clear that this set is exactly C.

It remains to show that the ultrapower of M by U is well-founded. It will help first to verify that if $C \in \mathfrak{X}$ codes a well-founded relation on κ, then C^* codes a well-founded relation on Ord^W. If $C \in \mathfrak{X}$ codes a well-founded relation, $\langle V_\kappa, \in, B\rangle_{B\in\mathfrak{X}}$ satisfies that $C \restriction \alpha$ has a rank function for all $\alpha < \kappa$. It follows that $\langle W, \in, B^*\rangle_{B\in\mathfrak{X}}$ satisfies that $C^* \restriction \alpha$ has a rank function for all $\alpha < \mathrm{Ord}^W$. Since κ is weakly compact and we assumed that W has size κ, we can find a *well-founded* elementary extension $\langle X, E, B^{**}\rangle_{B\in\mathfrak{X}}$ for the structure $\langle W, \in, B^*\rangle_{B\in\mathfrak{X}}$ satisfying that there exists an ordinal above the ordinals of W (Theorem 1.1(1)). There is no reason to expect that X is an end-extension or that E is the true membership relation, but that is not important for us. We only care that E is well-founded and X thinks it has an ordinal $> \mathrm{Ord}^W$. By elementarity, it follows that $\langle X, E, B^{**}\rangle_{B\in\mathfrak{X}}$ satisfies that $C^{**} \restriction \alpha$ has a rank function for all $\alpha < \mathrm{Ord}^X$. In particular, if $\alpha > \mathrm{Ord}^W$ in X, then $\langle X, E, B^{**}\rangle_{B\in\mathfrak{X}}$ satisfies that $C^{**} \restriction \alpha$ has a rank function. Since the structure $\langle X, E, B^{**}\rangle_{B\in\mathfrak{X}}$ is well-founded and can only add new elements to C^*, if C^* was not well-founded to begin with, X would detect this. Hence C^* is well-founded.

Now we go back to proving that the ultrapower of M by U is well-founded. Suppose towards a contradiction that there exists a membership descending sequence $\ldots E\,[f_n]\,E\ldots E\,[f_1]\,E\,[f_0]$ of elements of the ultrapower. Each $f_n:\kappa\to M$ is an element of M, and for every $n\in\omega$, the set $A_n=\{\alpha\in\kappa\mid f_{n+1}(\alpha)\in f_n(\alpha)\}\in U$. In M, we will define subsets E_n and F_n of κ coding information about f_n. Fix a transitive set T_n in M such that the range of f_n is contained in T_n and let E_n be a relation on κ coding T_n. Let F_n be a function on κ such that $F_n(\alpha)$ is the element representing $f_n(\alpha)$ in E_n. Next, we define subsets B_n of κ as follows. If $\alpha\in A_n$, then $f_{n+1}(\alpha)\in f_n(\alpha)$ and therefore the transitive closure of $f_{n+1}(\alpha)$ is a subset of the transitive closure of $f_n(\alpha)$. Thus, for $\alpha\in A_n$, there is a membership preserving map φ^n_α mapping a transitive subset of E_{n+1} containing $F_{n+1}(\alpha)$ onto a transitive subset of E_n such that $\varphi^n_\alpha(F_{n+1}(\alpha))$ is an E_n element of $F_n(\alpha)$. Let B_n code a collection of such maps φ^n_α for $\alpha\in A_n$.

By the observation above, each E^*_n codes a well-founded relation on κ^*. Since each $A_n\in U$, it follows that $\kappa\in A^*_n$ and so, by elementarity, B^*_n codes a membership preserving map φ^n_κ from a transitive subset of E^*_{n+1} to a transitive subset of E^*_n such that $\varphi^n_\kappa(F^*_{n+1}(\kappa))$ is an E^*_n element of $F^*_n(\kappa)$. If we let π_n be the Mostowski collapse of E^*_n, then by the uniqueness of the Mostowski collapse, we have that $\pi_{n+1}(F^*_{n+1}(\kappa))=\pi_n(\varphi^n_\kappa(F^*_{n+1}(\kappa)))$ and by the definition of φ^n_κ, we have that $\pi_n(\varphi^n_\kappa(F^*_{n+1}(\kappa)))$ is an element of $\pi_n(F^*_n(\kappa))$. It follows that the elements $\pi_n(F^*_n(\kappa))$ form a descending $\in$-sequence. Thus, we have reached a contradiction showing that the ultrapower of M by U is well-founded. □

4. Ramsey Cardinals

Recall that Theorem 1.3 gave a characterization of Ramsey cardinals in terms of the existence of elementary embeddings for weak κ-models of set theory. For convenience, we restate the theorem below.

Theorem. *A cardinal κ is Ramsey if and only if every $A\subseteq\kappa$ is contained in a weak κ-model M for which there exists a κ-powerset preserving elementary embedding $j:M\to N$ with the additional property that whenever $\langle A_n\mid n\in\omega\rangle$ are subsets of κ such that each $A_n\in M$ and $\kappa\in j(A_n)$, then $\cap_{n\in\omega}A_n\neq\emptyset$.*

Proposition 2.8(3) restated the characterization in terms of the existence of M-ultrafilters. Again, we restate it here for convenience.

Proposition. *A cardinal κ is Ramsey if and only if every $A\subseteq\kappa$ is contained in a weak κ-model M for which there exists a weakly amenable countably complete M-ultrafilter on κ.*

In Theorem 3.10, we proved the backward direction of Proposition 2.8(3). In this section, I give a much more involved proof of the forward direction, essentially following [Dod82]. I conclude the section with another embedding characterization for Ramsey and strongly Ramsey cardinals.

Definition 4.1. Suppose κ is a cardinal and $A\subseteq\kappa$. Then $I\subseteq\kappa$ is a *good set of indiscernibles* for $\langle L_\kappa[A],A\rangle$ if for all $\gamma\in I$:

(1) $\langle L_\gamma[A\cap\gamma],A\cap\gamma\rangle\prec\langle L_\kappa[A],A\rangle$.

(2) $I \setminus \gamma$ is a set of indiscernibles for $\langle L_\kappa[A], A, \xi\rangle_{\xi\in\gamma}$.

It turns out that if I is a good set of indiscernibles for $\langle L_\kappa[A], A\rangle$, then every $\gamma \in I$ is inaccessible in $L_\kappa[A]$. Thus, in particular, $L_\kappa[A]$ is the union of an elementary chain of models of ZFC, and hence is itself a model of ZFC.

Lemma 4.2. *If κ is Ramsey and $A \subseteq \kappa$, then $\langle L_\kappa[A], A\rangle$ has a good set of indiscernibles of size κ.*

For a proof, see [Dod82] (Ch. 17). We are now ready to prove that if κ is Ramsey, then every $A \subseteq \kappa$ can be put into a weak κ-model M for which there exists a weakly amenable countably complete M-ultrafilter on κ.

Proof of the forward direction of Proposition 2.8(3). Fix $A \subseteq \kappa$ and consider the structure $\mathcal{A} = \langle L_\kappa[A], A\rangle$. Note that $\mathcal{A}$ has definable Skolem functions since it has a definable well-ordering. By Lemma 4.2, $\mathcal{A}$ has a good set indiscernibles I of size κ. For every $\gamma \in I$ and $n \in \omega$, let $\vec{\gamma}_n$ denote the sequence $\gamma_1 < \gamma_2 < \cdots < \gamma_n$ of the first n indiscernibles in I above γ. Given $\gamma \in I$ and $n \in \omega$, let $\widetilde{\mathcal{M}}^n_\gamma = \langle \widetilde{M}^n_\gamma, A \cap \widetilde{M}^n_\gamma\rangle = Scl_{\mathcal{A}}(\gamma + 1 \cup \vec{\gamma}_n)$ be the Skolem closure using the definable Skolem functions of $\mathcal{A}$. Since $L_\kappa[A]$ is a model of ZFC, it satisfies that for every λ, H_λ exists and is a model of ZFC$^-$. Since $\widetilde{\mathcal{M}}^n_\gamma \prec \mathcal{A}$ and $\gamma \in \widetilde{M}^n_\gamma$, we have $H^{\mathcal{A}}_{\gamma^+} \in \widetilde{M}^n_\gamma$. Next, let $M^n_\gamma = \widetilde{M}^n_\gamma \cap H^{\mathcal{A}}_{\gamma^+}$ and $\mathcal{M}^n_\gamma = \langle M^n_\gamma, A \cap M^n_\gamma\rangle$.

Lemma 4.2.1. *Each M^n_γ is a transitive model of* ZFC$^-$.

Proof. It is clear that $M^n_\gamma \models$ ZFC$^-$ since it is precisely the H_{γ^+} of $\widetilde{\mathcal{M}}^n_\gamma$ and $\widetilde{\mathcal{M}}^n_\gamma$ knows by elementarity that H_{γ^+} satisfies ZFC$^-$.

To see that M^n_γ is transitive, fix $a \in M^n_\gamma$ and $b \in a$. The set a is an element of $H^{\mathcal{A}}_{\gamma^+}$ and is therefore coded by a subset of $\gamma \times \gamma$ in $\mathcal{A}$. By elementarity, $\widetilde{M}^n_\gamma$ contains a set $E \subseteq \gamma \times \gamma$ coding a and the Mostowski collapse $\pi : \langle \gamma, E\rangle \to Trcl(a)$. Let $\alpha \in \gamma$ such that $\mathcal{A} \models \pi(\alpha) = b$. Since $\alpha < \gamma$, we have $\alpha \in \widetilde{M}^n_\gamma$, and so $b \in \widetilde{M}^n_\gamma$ by elementarity. Since clearly $b \in H^{\mathcal{A}}_{\gamma^+}$, we have $b \in M^n_\gamma$. $\square$

Lemma 4.2.2. *For every $\gamma \in I$ and $n \in \omega$, we have $\mathcal{M}^n_\gamma \prec \mathcal{M}^{n+1}_\gamma$.*

Proof. It suffices to observe that $\widetilde{\mathcal{M}}^n_\gamma \prec \widetilde{\mathcal{M}}^{n+1}_\gamma$ together with the fact that $\mathcal{M}^n_\gamma = \langle H^{\widetilde{\mathcal{M}}^n_\gamma}_{\gamma^+}, A \cap H^{\widetilde{\mathcal{M}}^n_\gamma}_{\gamma^+}\rangle$ and $\mathcal{M}^{n+1}_\gamma = \langle H^{\widetilde{\mathcal{M}}^{n+1}_\gamma}_{\gamma^+}, A \cap H^{\widetilde{\mathcal{M}}^{n+1}_\gamma}_{\gamma^+}\rangle$. $\square$

Recall that if $a \in \widetilde{M}^n_\gamma$, then $a = h(\xi_0, \ldots, \xi_m, \gamma, \vec{\gamma}_n)$ where h is a definable Skolem function, $\xi_i \in \gamma$, and $\vec{\gamma}_n$ are the first n indiscernibles above γ in I. Given $\gamma < \delta \in I$, define $f^n_{\gamma\delta} : \widetilde{M}^n_\gamma \to \widetilde{M}^n_\delta$ by $f^n_{\gamma\delta}(a) = h(\xi_0, \ldots, \xi_m, \delta, \vec{\delta}_n)$ where $a = h(\xi_0, \ldots, \xi_m, \gamma, \vec{\gamma}_n)$ is as above. Observe that since $I \setminus \gamma$ are indiscernibles for $\langle L_\kappa[A], A, \xi\rangle_{\xi\in\gamma}$ (Definition 4.1(2)), the map $f^n_{\gamma\delta}$ is clearly well-defined. It is, moreover, an elementary embedding of the structure $\widetilde{\mathcal{M}}^n_\gamma$ into $\widetilde{\mathcal{M}}^n_\delta$. Since $f^n_{\gamma\delta}(\gamma) = \delta$ and $f^n_{\gamma\delta}(\xi) = \xi$ for all $\xi < \gamma$, the critical point of $f^n_{\gamma\delta}$ is γ. Finally, note that for all $\gamma < \delta < \beta \in I$, we have $f^n_{\gamma\beta} \circ f^n_{\beta\delta} = f^n_{\gamma\delta}$.

Lemma 4.2.3. *The map $f^n_{\gamma\delta} \restriction M^n_\gamma : M^n_\gamma \to M^n_\delta$ is an elementary embedding of the structure $\mathcal{M}^n_\gamma$ into $\mathcal{M}^n_\delta$.*

Proof. Fix $a \in M^n_\gamma$ and recall that $\tilde{M}{}^n_\gamma$ thinks $a \in H_{\gamma^+}$. By elementarity of the $f^n_{\gamma\delta}$, it follows that $\tilde{M}{}^n_\delta$ thinks $f^n_{\gamma\delta}(a) \in H_{\delta^+}$. Therefore $f^n_{\gamma\delta} : M^n_\gamma \to M^n_\delta$. Elementarity follows as in the previous lemma. □

For $\gamma \in I$, define $U^n_\gamma = \{X \in \mathcal{P}(\gamma)^{\mathcal{M}^n_\gamma} \mid \gamma \in f^n_{\gamma\delta}(X)$ for some $\delta > \gamma\}$. Equivalently, we could have used "for *all* $\delta > \gamma$" in the definition.

Lemma 4.2.4. *U^n_γ is an M^n_γ-ultrafilter on γ.*

Proof. Easy. □

Observe that if $a_0, \ldots, a_n \in L_\gamma[A]$ for some $\gamma \in I$, then for every formula $\varphi(\overrightarrow{x})$, we have $\langle L_\kappa[A], A\rangle \models \varphi(\overrightarrow{a}) \leftrightarrow \langle L_\gamma[A \cap \gamma], A \cap \gamma\rangle \models \varphi(\overrightarrow{a}) \leftrightarrow \langle L_\kappa[A], A\rangle \models$ "$\langle L_\gamma[A \cap \gamma], A \cap \gamma\rangle \models \varphi(\overrightarrow{a})$". It follows that for every $\gamma \in I$, the model $\langle L_\kappa[A], A\rangle$ has a truth predicate for formulas with parameters from $L_\gamma[A]$ that is definable with γ as a parameter.

Lemma 4.2.5. *The M^n_γ-ultrafilter U^n_γ is an element of M^{n+2}_γ.*

Proof. In $\overset{\sim}{\mathcal{M}}{}^{n+2}_\gamma$, we have $U^n_\gamma = \{x \in \mathcal{P}(\gamma) \mid \exists h \exists \xi_0, \ldots, \xi_m < \gamma\ h$ is a Skolem term and $h = (\xi_0, \ldots, \xi_m, \gamma, \gamma_1, \ldots, \gamma_n) \wedge \gamma \in h(\xi_0, \ldots, \xi_m, \gamma_1, \gamma_2, \ldots, \gamma_{n+1})\}$. This follows since $\gamma_{n+2} \in \tilde{M}{}^{n+2}_\gamma$, and therefore we can define a truth predicate for $L_{\gamma_{n+2}}[A]$, which is good enough for the definition above. So far we have shown that U^n_γ is in $\tilde{M}{}^{n+2}_\gamma$, but obviously $U^n_\gamma \in H^{\mathcal{A}}_{\gamma^+}$, and therefore $U^n_\gamma \in M^{n+2}_\gamma$. □

It should be clear that $U^n_\gamma \subseteq U^{n+1}_\gamma$ and $f^n_{\gamma\delta} \subseteq f^{n+1}_{\gamma\delta}$. Let $M_\gamma = \cup_{n\in\omega} M^n_\gamma$ and $\mathcal{M}_\gamma = \langle M_\gamma, A \cap M_\gamma\rangle$. Note that M_γ is a transitive model of ZFC$^-$. Let $U_\gamma = \cup_{n\in\omega} U^n_\gamma$ and $f_{\gamma\delta} = \cup_{n\in\omega} f^n_{\gamma\delta} : M_\gamma \to M_\delta$. The map $f_{\gamma\delta}$ is an elementary embedding from the structure $\mathcal{M}_\gamma$ into $\mathcal{M}_\delta$ and U_γ is an M_γ-ultrafilter on γ.

Lemma 4.2.6. *The M_γ-ultrafilter U_γ is weakly amenable.*

Proof. Fix a sequence $\langle B_\alpha \mid \alpha \in \gamma\rangle$ in M_γ of subsets of γ. We need to show that $C = \{\xi \in \gamma \mid B_\xi \in U_\gamma\}$ is an element of M_γ. Since $\langle B_\alpha \mid \alpha < \kappa\rangle \in M_\gamma$, it follows that $\langle B_\alpha \mid \alpha < \kappa\rangle \in M^n_\gamma$ for some $n \in \omega$. But then $C = \{\xi \in \gamma \mid B_\xi \in U^n_\gamma\}$ and $U^n_\gamma \in M^{n+2}_\gamma \subseteq M_\gamma$. □

Now for every $\gamma \in I$, we have an associated structure $\langle M_\gamma, \in, A \cap M_\gamma, U_\gamma\rangle$. Also, if $\gamma < \delta$ in I, we have an elementary embedding $f_{\gamma\delta} : M_\gamma \to M_\delta$ with critical point γ between the structures $\mathcal{M}_\gamma$ and $\mathcal{M}_\delta$ such that $X \in U_\gamma$ if and only if $f_{\gamma\delta}(X) \in U_\delta$. This is a directed system of embeddings, and so we can take its direct limit. Define $\langle B, E, A', V\rangle = \lim_{\gamma\in I}\langle M_\gamma, \in, A \cap M_\gamma, U_\gamma\rangle$.

Lemma 4.2.7. *The relation E on B is well-founded.*

Proof. The elements of B are functions t with domains $\{\xi \in I \mid \xi \geq \alpha\}$ for some $\alpha \in I$ satisfying the properties:

(1) $t(\gamma) \in M_\gamma$,

(2) for $\gamma < \delta$ in domain of t, we have $t(\delta) = f_{\gamma\delta}(t(\gamma))$,
(3) there is no $\xi \in I \cap \alpha$ for which there is $a \in M_\xi$ such that $f_{\xi\alpha}(a) = t(a)$.

Note that each t is determined once you know any $t(\xi)$ by extending uniquely forward and backward. Standard arguments (for example, [Jec03], Ch. 12) show that $\langle B, E, A'\rangle \models \varphi(t_1, \ldots, t_n) \leftrightarrow \exists\gamma\, \mathcal{M}_\gamma \models \varphi(t_1(\gamma), \ldots, t_n(\gamma)) \leftrightarrow$ for all γ in the intersection of the domains of the t_i, the structure $\mathcal{M}_\gamma \models \varphi(t_1(\gamma), \ldots, t_n(\gamma))$. Note that this truth definition holds only of atomic formulas where the formulas involve the predicate for the ultrafilter.

Suppose to the contrary that E is not well-founded, then there is a descending E-sequence $\cdots E\, t_n\, E \cdots E\, t_1\, E\, t_0$. Find γ_0 such that $\mathcal{M}_{\gamma_0} \models t_1(\gamma_0) \in t_0(\gamma_0)$. Next, find $\gamma_1 > \gamma_0$ such that $\mathcal{M}_{\gamma_1} \models t_2(\gamma_1) \in t_1(\gamma_1)$. In this fashion, define an increasing sequence $\gamma_0 < \gamma_1 < \cdots < \gamma_n < \cdots$ such that $\mathcal{M}_{\gamma_n} \models t_{n+1}(\gamma_n) \in t_n(\gamma_n)$. Let $\gamma \in I$ such that $\gamma > \sup_{n\in\omega}\gamma_n$. It follows that for all $n \in \omega$, the structure $\mathcal{M}_\gamma \models f_{\gamma_n\gamma}(t_{n+1}(\gamma_n)) \in f_{\gamma_n\gamma}(t_n(\gamma_n))$, and therefore $\mathcal{M}_\gamma \models t_{n+1}(\gamma) \in t_n(\gamma)$. But, of course, this is impossible. Thus, E is well-founded. □

Let $\langle M, \in, A^*, U\rangle$ be the Mostowski collapse of $\langle B, E, A', V\rangle$.

Lemma 4.2.8. *The cardinal κ is an element of M.*

Proof. Fix $\alpha \in \kappa$ and let γ be the least ordinal in I above α. Let t_α have domain $\{\xi \in I \mid \xi \geq \gamma\}$ with $t_\alpha(\xi) = \alpha$. The function t_α is an element of B which collapses to α. Let t_κ have domain I with $t_\kappa(\gamma) = \gamma$. The function t_κ is an element of B which collapses to κ. □

Let $j_\gamma : M_\gamma \to M$ such that $j_\gamma(a)$ is the collapse of the function t for which $t(\gamma) = a$. The map j_γ is an elementary embedding of the structure $\mathcal{M}_\gamma$ into $\langle M, \in, A^*\rangle$. It is, moreover, elementary for atomic formulas in the language with the predicate for the ultrafilter. Observe that $j_\gamma(\xi) = \xi$ for all $\xi < \gamma$ since if $t(\gamma) = \xi$, then $t = t_\xi$. Also, $j_\gamma(\gamma) = \kappa$ since if $t(\gamma) = \gamma$, then $t = t_\kappa$. So the critical point of each j_γ is κ. Finally, if $\gamma < \delta$ in I, then $j_\delta \circ f_{\gamma\delta} = j_\gamma$.

Lemma 4.2.9. *The set U is a weakly amenable M-ultrafilter on κ.*

Proof. Easy. □

Lemma 4.2.10. *A set $X \in U$ if and only if there exists $\alpha \in I$ such that $\{\xi \in I \mid \xi > \alpha\} \subseteq X$.*

Proof. Fix $X \subseteq \kappa$ in M and $\beta \in I$ such that for all $\xi > \beta$, there is $X' \in M_\xi$ with $j_\xi(X') = X$. For $\xi > \beta$, we have $X \in U \leftrightarrow X' \in U_\xi \leftrightarrow \xi \in f_{\xi\xi_1}(X') \leftrightarrow j_{\xi_1}(\xi) \in j_{\xi_1} \circ f_{\xi\xi_1}(X') = j_\xi(X') \leftrightarrow \xi \in j_\xi(X') = X$. Thus, for $\alpha > \beta$, we have $\{\xi \in I \mid \xi > \alpha\} \subseteq X$ if and only if $X \in U$. □

Lemma 4.2.11. *The M-ultrafilter U is countably complete.*

Proof. Fix $\langle A_n \mid n \in \omega\rangle$ a sequence of elements of U. We need to show that $\cap_{n\in\omega} A_n \neq \emptyset$. For each A_n, there exists $\gamma_n \in I$ such that $X_n = \{\xi \in I \mid \xi > \gamma_n\} \subseteq A_n$. Thus, $\cap_{n\in\omega} X_n \subseteq \cap_{n\in\omega} A_n$ and clearly $\cap_{n\in\omega} X_n$ has size κ. □

It remains to show that $A \in M$.

Lemma 4.2.12. *The set $A^* \restriction \kappa = A$, and hence $A \in M$.*

Proof. Fix $\alpha \in A$ and let $\gamma \in I$ such that $\gamma > \alpha$, then $\mathcal{M}_\gamma \models \alpha \in A$. It follows that $\langle M, \in, A^*\rangle \models j_\gamma(\alpha) \in A^*$, but $j_\gamma(\alpha) = \alpha$, and so $\alpha \in A^*$. Thus, $A \subseteq A^*$. Now fix $\alpha \in A^* \upharpoonright \kappa$ and let $\gamma \in I$ such that $\gamma > \alpha$, then $j_\gamma(\alpha) = \alpha$, and so $j_\gamma(\alpha) \in A^*$. It follows that $\alpha \in A$. Thus, $A^* \upharpoonright \kappa \subseteq A$. We conclude that $A = A^* \upharpoonright \kappa$. □

To summarize, we have shown that M is a weak κ-model since it is a transitive model of ZFC$^-$ of size κ containing κ as an element. We have further shown that A is an element of M and U is a countably complete weakly amenable M-ultrafilter. □

I conclude this section with some basic facts about finite products and iterations of weakly amenable countably complete M-ultrafilters.

Lemma 4.3. *Suppose M is a weak κ-model, U is a 1-good M-ultrafilter on κ, and $j : M \to N$ is the ultrapower by U. Then $j_U(U) = \{A \subseteq j_U(\kappa) \mid A = [f]_U$ and $\{\alpha \in \kappa \mid f(\alpha) \in U\} \in U\}$ is a weakly amenable N-ultrafilter on $j_U(\kappa)$ containing $j''U$ as a subset.*

For proof, see [Kan03] (Ch. 4, Sec. 19). Lemma 4.3 is essentially saying that we can take the ultrapower of the structure $\langle M, \in, U\rangle$ by the M-ultrafilter U and the Łoś Theorem still goes through due to the weak amenability of U. Previously, we only took ultrapowers of the structures $\langle M, \in\rangle$. The Łoś Theorem there relied on the fact that $M \models$ ZFC$^-$, but the structure $\langle M, \in, U\rangle$ need not satisfy any substantial fragment of ZFC. Thus, the additional assumption of weak amenablity is precisely what is required to carry out the argument. The ultrafilter $j_U(U)$ is simply the relation corresponding to U in the ultrapower.

The next lemma is an adaptation to the case of weak κ-models of the standard fact from iterating ultrapowers.

Lemma 4.4. *Suppose M is a weak κ-model, U is a 1-good M-ultrafilter on κ, and $j_U : M \to M/U$ is the ultrapower by U. Suppose further that*

$$j_{U^n} : M \to M/U^n \text{ and } h_{U^n} : M/U \to (M/U)/U^n$$

are the well-founded ultrapowers by U^n. Then the ultrapower

$$j_{j_{U^n}(U)} : M/U^n \to (M/U^n)/j_{U^n}(U) \text{ (by } j_{U^n}(U))$$

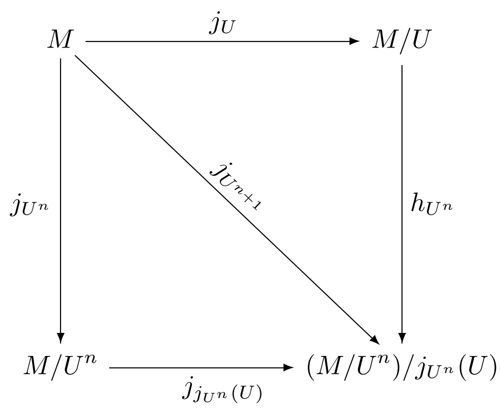
and the ultrapower

$$j_{U^{n+1}} : M \to M/U^{n+1} \text{ (by } U^{n+1})$$

are also well-founded. Moreover,

$$(M/U^n)/j_{U^n}(U) = (M/U)/U^n = M/U^{n+1},$$

and the following diagram commutes:



Proof. The idea is to define obvious isomorphisms between $(M/U^n)/j_{U^n}(U)$ and M/U^{n+1} and between $(M/U)/U^n$ and M/U^{n+1}. □

Proposition 4.5. *If M is a weak κ-model and U is a weakly amenable countably complete M-ultrafilter on κ, then the ultrapowers of M by U^n are well-founded for all $n \in \omega$.*

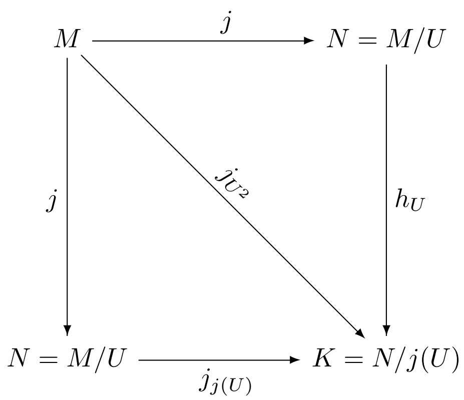
Proof. Use Lemma 4.4 and argue by induction on n. □

The commutative diagram above gives an interesting reformulation of the embeddings for Ramsey and strongly Ramsey cardinals.

Proposition 4.6. *A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is contained in a weak κ-model $M \models$ ZFC for which there exists a weakly amenable countably complete M-ultrafilter with the ultrapower $j : M \to N$ having $M \prec N$.*

The difference from the earlier embeddings is that now for every $A \subseteq \kappa$, we have $A \in M$ where M is a model of full ZFC and $j : M \to N$ is an embedding such that not only do M and N have the same subsets of κ, but actually $M \prec N$.

Proof. Fix $A \subseteq \kappa$ and choose a weak κ-model M containing A and V_κ for which there exists a weakly amenable countably complete M-ultrafilter U on κ. Let $j : M \to N$ be the ultrapower by U. The commutative diagram from Lemma 4.4 becomes the following for the case $n = 1$:



Let $M' = V^N_{j(\kappa)}$ and observe that it is a transitive model of ZFC. Let $K' = V^K_{j_{j(U)}(j(\kappa))} = V^K_{h_U(j(\kappa))}$. Since $j(\kappa)$ is regular in N, the models M' and N have the same functions from κ to M', and thus the map $h_U \restriction M' : M' \to K'$ is the ultrapower embedding of M' into M'/U. It remains to show that $M' \prec K'$, but this follows easily from the $j_{j(U)}$ side of the commutative diagram since $j_{j(U)} \restriction M' : M' \to K'$ and $j_{j(U)}$ is identity on M'. □

Corollary 4.7. *A cardinal κ is strongly Ramsey if and only if every $A \subseteq \kappa$ is contained in a κ-model $M \models$ ZFC for which there exists an elementary embedding $j : M \to N$ with critical point κ such that $M \prec N$.*

Proof. Using the previous proof it suffices to verify that M' is a κ-model, that is, it is closed under $< \kappa$-sequences. By Proposition 2.3, we can assume, without loss of generality, that N is closed under $< \kappa$-sequences. Since N thinks that $j(\kappa)$ is inaccessible, it follows that $M' = V^N_{j(\kappa)}$ must be closed under $< \kappa$-sequences as well. □

The same arguments cannot be carried out for weakly Ramsey cardinals since the ultrapower by U^2 need not be well-founded in that case. Thus, it is not clear whether the weakly Ramsey cardinals have a similar characterization.

Question 4.8. If κ is a weakly Ramsey cardinal, does there exist for every $A \subseteq \kappa$, a weak κ-model $M \models$ ZFC and an elementary embedding $j : M \to N$ with critical point κ such that $M \prec N$?

5. α-iterable Cardinals and α-good Ultrafilters

A key feature of M-ultrafilters associated with Ramsey-like embeddings is weak amenability; meaning that they have the potential to be iterated. Recall that 0-good M-ultrafilters are exactly the ones with well-founded ultrapowers, and 1-good M-ultrafilters are weakly amenable and 0-good. Consider starting with a weak κ-model M_0 for which there exists a 1-good M_0-ultrafilter U_0. Applying Lemma 4.3, take the ultrapower of the structure $\langle M_0, \in, U_0\rangle$ by U_0 to obtain the structure $\langle M_1, \in, U_1\rangle$ with U_1 a weakly amenable M_1-ultrafilter. Next, if the result happens to be *well-founded*, take the ultrapower of $\langle M_1, \in, U_1\rangle$ by U_1 to obtain the structure $\langle M_2, \in, U_2\rangle$. We will call such U_0 2-*good* to indicate that we were able to do a two-step iteration. If $\xi \leq \omega$ and we can continue iterating for ξ-many steps by obtaining well-founded ultrapowers, we will say that the M-ultrafilter U_0 is ξ-*good*. Suppose that U_0 is ω-good. In this case, we have a directed system of models $\langle M_n, \in, U_n\rangle$ with the corresponding ultrapower embeddings. Take the direct limit of this system. If the direct limit happens to be well-founded, collapse it to obtain $\langle M_\omega, \in, U_\omega\rangle$, where U_ω is a weakly amenable M_ω-ultrafilter. We will call U_0 $\omega{+}1$-*good* to indicate that we were able to do an $\omega + 1$-step iteration. We can proceed in this fashion as long as the iterates are well-founded.[14]

Definition 5.1. Suppose M is a weak κ-model. An M-ultrafilter on κ is α-*good*, if the ultrapower construction can be iterated α-many steps.

Definition 5.2. A cardinal κ is α-*iterable* if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists an α-good M-ultrafilter on κ.

[14] See [Kan03] (Ch. 4, Sec. 19) for details involved in this construction.

Weakly Ramsey cardinals are exactly the 1-iterable cardinals. Gaifman showed in [Gai74] that if an M-ultrafilter is ω_1-good, then it is already α-good for every ordinal α. Kunen showed in [Kun70] that if an M-ultrafilter is weakly amenable and countably complete, then it is ω_1-good. Thus, Ramsey cardinals are ω_1-iterable. Even though countable completeness is sufficient for full iterability, it is not necessary. In fact, Sharpe and Welch showed in [WS10] that ω_1-iterable cardinals are strictly weaker than Ramsey cardinals.

Theorem 5.3. *An ω_1-Erdős is a limit of ω_1-iterable cardinals.*

In an upcoming paper with Welch [GW10], we show:

Theorem 5.4. *The α-iterable cardinals forms a strict hierarchy for $\alpha \leq \omega_1$. In particular, for $\alpha < \beta \leq \omega_1$, a β-iterable cardinal is a limit of α-iterable cardinals.*

Theorem 5.5. *The α-iterable cardinals are downwards absolute to L for $\alpha < \omega_1^L$.*

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