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# RAMSEY-LIKE CARDINALS II

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Abstract. This paper continues the study of the *Ramsey-like* large cardinals introduced in [Git09] and [WS10]. Ramsey-like cardinals are defined by generalizing the characterization of Ramsey cardinals via the existence of elementary embeddings. Ultrafilters derived from such embeddings are fully iterable and so it is natural to ask about large cardinal notions asserting the existence of ultrafilters allowing only $\alpha$-many iterations for some countable ordinal $\alpha$. Here we study such *$\alpha$-iterable* cardinals. We show that the $\alpha$-iterable cardinals form a strict hierarchy for $\alpha \leq \omega_1$, that they are downward absolute to $L$ for $\alpha < \omega_1^L$, and that the consistency strength of Schindler's remarkable cardinals is strictly between 1-iterable and 2-iterable cardinals.

We show that the strongly Ramsey and super Ramsey cardinals from [Git09] are downward absolute to the core model $K$. Finally, we use a forcing argument from a strongly Ramsey cardinal to separate the notions of Ramsey and *virtually Ramsey* cardinals. These were introduced in [WS10] as an upper bound on the consistency strength of the Intermediate Chang's Conjecture.

## 1. Introduction

The definitions of measurable cardinals and stronger large cardinal notions follow the template of asserting the existence of elementary embeddings $j : V \to M$ from the universe of sets to a transitive subclass with that cardinal as the critical point. Many large cardinal notions below a measurable cardinal can be characterized by the existence of elementary embeddings as well. The characterizations of these smaller large cardinals $\kappa$ follow the template of asserting the existence of elementary embeddings $j : M \to N$ with critical point $\kappa$ from a weak $\kappa$-model or $\kappa$-model $M$ of set theory to a transitive set.[1] A *weak $\kappa$-model* $M$ of set theory is a transitive set of size $\kappa$ satisfying ZFC$^-$ (ZFC without the *Powerset Axiom*) and having $\kappa \in M$. If a weak $\kappa$-model $M$ is additionally closed under $<\kappa$-sequences, that is $M^{<\kappa} \subseteq M$, it is called a *$\kappa$-model* of set theory. Having embeddings on $\kappa$-models is particularly important for forcing indestructibility arguments, where the techniques rely on $<\kappa$-closure. The weakly compact cardinal is one example of a smaller large cardinal that is characterized by the existence of elementary embeddings. A cardinal $\kappa$ is *weakly compact* if $\kappa^{<\kappa} = \kappa$ and every $A \subseteq \kappa$ is contained in a weak $\kappa$-model $M$ for which there exists an elementary embedding $j : M \to N$ with critical point $\kappa$. Another example is the strongly unfoldable cardinal. A cardinal $\kappa$ is *strongly unfoldable* if for every ordinal $\alpha$, every $A \subseteq \kappa$ is contained a weak $\kappa$-model $M$ for which there exists an elementary embedding $j : M \to N$ with critical point $\kappa$, $\alpha < j(\kappa)$, and $V_\alpha \subseteq N$.

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[1] It will be assumed throughout the paper that, unless stated otherwise, all embeddings are elementary and between transitive structures.

An embedding $j : V \to M$ with critical point $\kappa$ can be used to construct a $\kappa$-complete ultrafilter on $\kappa$. These measures are *fully iterable*; they allow iterating the ultrapower construction through all the ordinals. The iteration proceeds by taking ultrapowers by the image of the original ultrafilter at successor ordinal stages and direct limits at limit ordinal stages to obtain a directed system of elementary embeddings of *well-founded* models of length $Ord$. Returning to smaller large cardinals, an embedding $j : M \to N$ with critical point $\kappa$ and $M$ a model of $\text{ZFC}^-$ can be used to construct an ultrafilter on $\mathcal{P}(\kappa)^M$ that is $\kappa$-complete *from the perspective* of $M$. These small measures are called *$M$-ultrafilters* because all their measure-like properties hold only from the perspective of $M$. It is natural to ask what kind of iterations can be obtained from $M$-ultrafilters. Here, there are immediate technical difficulties arising from the fact that an $M$-ultrafilter is, in most interesting cases, external to $M$. In order to start defining the iteration, the $M$-ultrafilter needs to have the additional property of being *weakly amenable*. The existence of weakly amenable $M$-ultrafilters on $\kappa$ with well-founded ultrapowers is equivalent to the existence of embeddings $j : M \to N$ with critical point $\kappa$ where $M$ and $N$ have the *same* subsets of $\kappa$. We will call such embeddings *$\kappa$-powerset preserving*.

Gitman observed in [Git09] that weakly compact cardinals are not strong enough to imply the existence of $\kappa$-powerset preserving embeddings. She called a cardinal $\kappa$ *weakly Ramsey* if every $A \subseteq \kappa$ is contained in a weak $\kappa$-model for which there exists a $\kappa$-powerset preserving elementary embedding $j : M \to N$. In terms of consistency strength weakly Ramsey cardinals are above completely ineffable cardinals and therefore much stronger than weakly compact cardinals. We associate iterating $M$-ultrafilters with Ramsey cardinals because Ramsey cardinals imply the existence of fully iterable $M$-ultrafilters. Mitchell showed in [Mit79] that $\kappa$ is Ramsey if and only if every $A \subseteq \kappa$ is contained in a weak $\kappa$-model $M$ for which there exists a weakly amenable countably complete[2] $M$-ultrafilter on $\kappa$. Kunen showed in [Kun70] that countable completeness is a sufficient condition for an $M$-ultrafilter to be fully iterable, that is, for every stage of the iteration to produce a well-founded model. The $\alpha$-iterable cardinals were introduced in [Git09] to fill the gap between weakly Ramsey cardinals that merely assert the existence of $M$-ultrafilters with the potential to be iterated and Ramsey cardinals that assert the existence of fully iterable $M$-ultrafilters. A cardinal $\kappa$ is *$\alpha$-iterable* if every subset of $\kappa$ is contained in a weak $\kappa$-model $M$ for which there exists an $M$-ultrafilter on $\kappa$ allowing an iteration of length $\alpha$. By a well-known result of Gaifman [Gai74], an ultrafilter that allows an iteration of length $\omega_1$ is fully iterable. So it only makes sense to study the $\alpha$-iterable cardinals for $\alpha \leq \omega_1$.

Welch and Sharpe showed in [WS10] that $\omega_1$-iterable cardinals are strictly weaker than $\omega_1$-Erdős cardinals. In Section 3, we show that for $\alpha < \omega_1^L$, the $\alpha$-iterable cardinals are downward absolute to $L$. In Section 4, we show that for $\alpha \leq \omega_1$, the $\alpha$-iterable cardinals form a hierarchy of strength. Also, in Section 4, we establish a relationship between $\alpha$-iterable cardinals and $\alpha$-Erdős cardinals, and provide an improved upper bound on the consistency strength of Schindler's remarkable cardinals by placing it strictly between 1-iterable cardinals and 2-iterable cardinals.

[2]An $M$-ultrafilter is *countably complete* if every countable collection of sets in the ultrafilter has a nonempty intersection (see Section 2).

Finally we answer a question of Gitman about whether 1-iterable cardinals imply existence of embeddings on weak $\kappa$-models of ZFC.

Gitman also introduced *strongly Ramsey* cardinals and *super Ramsey* cardinals by requiring the existence of $\kappa$-powerset preserving embeddings on $\kappa$-models instead of weak $\kappa$-models. The strongly Ramsey and super Ramsey cardinals fit in between Ramsey cardinals and measurable cardinals in strength. In Section 5, we show that these two large cardinal notions are downward absolute to the core model $K$. In Section 6, we use a forcing argument starting from a strongly Ramsey cardinal to separate the notions of virtually Ramsey and Ramsey cardinals. *Virtually Ramsey* cardinals were introduced by Welch and Sharpe in [WS10] as an upper bound on the consistency of the Intermediate Chang's Conjecture.

## 2. Preliminaries

In this section, we review facts about $M$-ultrafilters and formally define the $\alpha$-iterable cardinals. We begin by giving a precise definition of an $M$-ultrafilter.

**Definition 2.1.** Suppose $M$ is a transitive model of $\mathrm{ZFC}^-$ and $\kappa$ is a cardinal in $M$. A set $U \subseteq \mathcal{P}(\kappa)^M$ is an *$M$-ultrafilter* if $\langle M, \in, U\rangle \models$ "$U$ is a $\kappa$-complete normal ultrafilter".

Recall that an ultrafilter is *$\kappa$-complete* if the intersection of any $<\kappa$-sized collection of sets in the ultrafilter is itself an element of the ultrafilter. An ultrafilter is *normal* if every function regressive on a set in the ultrafilter is constant on a set in the ultrafilter. By definition, $M$-ultrafilters are $\kappa$-complete and normal only from the *point of view* of $M$, that is, the collection of sets being intersected or the regressive function has to be an element of $M$. We will say that an $M$-ultrafilter is *countably complete* if every countable collection of sets in the ultrafilter has a nonempty intersection. Obviously, any $M$-ultrafilter is, by definition, countably complete from the point of view of $M$, but countable completeness requests the property to hold of *all* sequences, not just those in $M$.[3] Unless $M$ satisfies some extra condition, such as being closed under countable sequences, an $M$-ultrafilter need not be countably complete. In this article we shall consider the usual ultrapower of a structure $M$ taken using only functions in $M$. We are thus not using fine-structural ultrapowers in our arguments. An ultrapower by an $M$-ultrafilter is not necessarily well-founded. An $M$-ultrafilter with a well-founded ultrapower may be obtained from an elementary embedding $j : M \to N$.

**Proposition 2.2.** *Suppose $M$ is a weak $\kappa$-model and $j : M \to N$ is an elementary embedding with critical point $\kappa$, then $U = \{A \in \mathcal{P}(\kappa)^M \mid \kappa \in j(A)\}$ is an $M$-ultrafilter on $\kappa$ with a well-founded ultrapower.*

In this case, we say that $U$ is *generated by $\kappa$ via $j$*. The well-foundedness of the ultrapower follows since it embeds into $N$.

To define $\alpha$-iterable cardinals, we will need the corresponding key notion of *$\alpha$-good* $M$-ultrafilters.

[3] It is more standard for countable completeness to mean $\omega_1$-completeness which requires the intersection to be an element of the ultrafilter. However, the weaker notion we use here is better suited to $M$-ultrafilters because the countable collection itself can be external to $M$, and so there is no reason to suppose the intersection to be an element of $M$.

**Definition 2.3.** Suppose $M$ is a weak $\kappa$-model. An $M$-ultrafilter $U$ on $\kappa$ is 0-*good* if the ultrapower of $M$ by $U$ is well-founded.

To begin discussing the iterability of $M$-ultrafilters, we need the following key definitions.

**Definition 2.4.** Suppose $M$ is a weak $\kappa$-model. An $M$-ultrafilter $U$ on $\kappa$ is *weakly amenable* if for every $A \in M$ of size $\kappa$ in $M$, the intersection $U \cap A$ is an element of $M$.

**Definition 2.5.** Suppose $M$ is a model of ZFC$^-$. An elementary embedding $j : M \to N$ with critical point $\kappa$ is *$\kappa$-powerset preserving* if $M$ and $N$ have the same subsets of $\kappa$.

It turns out that the existence of weakly amenable 0-good $M$-ultrafilters on $\kappa$ is equivalent to the existence of $\kappa$-powerset preserving embeddings.

**Proposition 2.6.** *Suppose $M$ is a transitive model of* ZFC$^-$.

(1) *If $j : M \to N$ is the ultrapower by a weakly amenable $M$-ultrafilter on $\kappa$, then $j$ is $\kappa$-powerset preserving.*
(2) *If $j : M \to N$ is a $\kappa$-powerset preserving embedding, then the $M$-ultrafilter $U = \{A \in \mathcal{P}(\kappa)^M \mid \kappa \in j(A)\}$ is weakly amenable.*

**Definition 2.7.** Suppose $M$ is a weak $\kappa$-model. An $M$-ultrafilter on $\kappa$ is 1-*good* if it is 0-good and weakly amenable.

**Lemma 2.8.** *Suppose $M$ is a weak $\kappa$-model, $U$ is a 1-good $M$-ultrafilter on $\kappa$, and $j : M \to N$ is the ultrapower by $U$. Define*

$$j(U) = \{A \in \mathcal{P}(j(\kappa))^N \mid A = [f] \text{ and } \{\alpha \in \kappa \mid f(\alpha) \in U\} \in U\}.$$

*Then $j(U)$ is a weakly amenable $N$-ultrafilter on $j(\kappa)$ containing $j''U$ as a subset.*

See [Kan03] for details on the above facts. Lemma 2.8 is essentially saying that the weak amenability of $U$ implies a partial Łoś Theorem for the ultrapower of $\langle M, \in, U\rangle$ by $U$ resulting in $j(U)$, the predicate corresponding to $U$ in the ultrapower, having the requisite properties. The resulting ultrapower is fully elementary in the language without the predicate for $U$ and $\Sigma_0$-elementary with the predicate. This suffices since the main purpose in taking the ultrapower in the extended language is to obtain the next ultrafilter in the iteration. Weak amenability serves as the basis of any fine structural analysis of measures and extenders [Zem02]. As we shall see later, it is not necessarily the case that the ultrapower by $j(U)$ is well-founded.

Suppose $M$ is a weak $\kappa$-model and $U_0$ is a 1-good $M$-ultrafilter on $\kappa$. Let $j(U_0) = U_1$ be the weakly amenable ultrafilter obtained as above for the ultrapower of $M$ by $U$. If the ultrapower by $U_1$ happens to be well-founded, we will say that $U_0$ is 2-*good*. In this way, we can continue iterating the ultrapower construction so long as the ultrapowers are well-founded. For $\xi \leq \omega$, we will say that $U$ is $\xi$-*good* if the first $\xi$-many ultrapowers are well-founded. Suppose next that the first $\omega$-many ultrapowers are well-founded. We can form their direct limit and ask if that is well-founded as well. If the direct limit of the first $\omega$-many iterates turns out to be well-founded, we will say that $U$ is $\omega + 1$-*good*. Continuing the pattern, we make the following definition.

**Definition 2.9.** Suppose $M$ is a weak $\kappa$-model and $\alpha$ is an ordinal. An $M$-ultrafilter on $\kappa$ is *$\alpha$-good*, if we can iterate the ultrapower construction for $\alpha$-many steps.

Gaifman showed in [Gai74] that to be able to iterate the ultrapower construction through all the ordinals it suffices to know that we can iterate through all the countable ordinals.

**Theorem 2.10.** *Suppose $M$ is a weak $\kappa$-model. An $\omega_1$-good $M$-ultrafilter is $\alpha$-good for every ordinal $\alpha$.*

Thus, the study of $\alpha$-good ultrafilters only makes sense for $\alpha \leq \omega_1$.

**Definition 2.11.** For $\alpha \leq \omega_1$, a cardinal $\kappa$ is *$\alpha$-iterable* if every $A \subseteq \kappa$ is contained in a weak $\kappa$-model $M$ for which there exists an $\alpha$-good $M$-ultrafilter on $\kappa$.

A few easy observations about the definition are in order.

**Remark 2.12.**

(1) If $\kappa^{<\kappa} = \kappa$, then $\kappa$ is 0-iterable if and only if $\kappa$ is weakly compact. Without the extra assumption $\kappa^{<\kappa} = \kappa$, being 0-iterable is not necessarily a large cardinal notion. Hamkins showed in [Ham07] that it is consistent for $2^\omega$ to be 0-iterable.
(2) Weakly Ramsey cardinals are exactly the 1-iterable cardinals. Unlike 0-iterability, 1-iterability implies inaccessibility and hence weak compactness (see [Git09] for the strength of 1-iterable cardinals).
(3) By our previous comments, Ramsey cardinals are $\omega_1$-iterable.
(4) $\omega_1$-iterable cardinals are *strongly unfoldable* in $L$ (see [Vil98]).

## 3. $\alpha$-ITERABLE CARDINALS IN $L$

In this section, we show that for $\alpha < \omega_1^L$, the $\alpha$-iterable cardinals are downward absolute to $L$. This result is optimal since $\omega_1$-iterable cardinals cannot exist in $L$.

Many of our arguments below will use the following two simple facts about weak $\kappa$-models.

**Remark 3.1.**

(1) If $M$ is a weak $\kappa$-model of height $\alpha$, then $L^M = L_\alpha$. Note that $M \cap L$ can be a proper superset of $L_\alpha$. That is, $M$ might contain constructible elements that it does not realize are constructible.
(2) If $M$ is a weak $\kappa$-model, $j : M \to N$ is an elementary embedding with critical point $\kappa$, and $X$ has size $\kappa$ in $M$, then $j \restriction X$ is an element of $N$. This follows since $j \restriction X$ is definable from an enumeration $f$ of $X$ in $M$ together with $j(f)$, both of which are elements of $N$.

Next, we give an argument why $\omega_1$-iterable cardinals cannot exist in $L$.

**Proposition 3.2.** *If there is an $\omega_1$-iterable cardinal, then $0^\#$ exists.*

*Proof.* Suppose $\kappa$ is an $\omega_1$-iterable cardinal. Fix a weak $\kappa$-model $M$ and an $\omega_1$-good $M$-ultrafilter $U$ on $\kappa$. By Theorem 2.10, $U$ is fully iterable. Let $j_\alpha : M_\alpha \to N_\alpha$ be the $\alpha^{\text{th}}$-iterated ultrapower of $U$. Observe that $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^{M_\alpha}$ for all $\alpha$.

By remark 3.1 (1), $M_\alpha \cap L$ contains $L_\alpha$. Thus, for a large enough $\alpha$, we have $\mathcal{P}(\kappa)^L \subseteq L_\alpha \subseteq M_\alpha$. It follows that $L_{\kappa^+} \subseteq M$. Thus, $j_\alpha$ restricts to an embedding on $L_{\kappa^+}$ and hence $0^\#$ exists. □

We will first show that if $0^\#$ exists, then the Silver indiscernibles are $\alpha$-iterable in $L$ for all $\alpha < \omega_1^L$. Later, we will modify this argument to show that for $\alpha < \omega_1^L$, the $\alpha$-iterable cardinals are downward absolute to $L$. We begin with the case of 1-iterable cardinals. We will make use of a standard lemma below (see [BJW82] for a proof).

**Lemma 3.3.** *If $0^\#$ exists and $\kappa$ is a Silver indiscernible, then* $\mathrm{cf}^V((\kappa^+)^L) = \omega$.

**Theorem 3.4.** *If $0^\#$ exists, then the Silver indiscernibles are 1-iterable in $L$.*

*Proof.* Let $I = \{i_\xi \mid \xi \in Ord\}$ be the Silver indiscernibles enumerated in increasing order. Fix $\kappa \in I$ and let $\lambda = (\kappa^+)^L$. Define $j : I \to I$ by $j(i_\xi) = i_\xi$ for all $i_\xi < \kappa$ and $j(i_\xi) = i_{\xi+1}$ for all $i_\xi \geq \kappa$ in $I$. The map $j$ extends, via the Skolem functions, to an elementary embedding $j : L \to L$ with critical point $\kappa$. Restrict to $j : L_\lambda \to L_{j(\lambda)}$, which is clearly $\kappa$-powerset preserving. Let $U$ be the weakly amenable $L_\lambda$-ultrafilter generated by $\kappa$ via $j$ as in Proposition 2.2. Since every $\alpha < \lambda$ has size $\kappa$ in $L_\lambda$, by weak amenability, $U \cap L_\alpha$ is an element of $L_\lambda$. Construct, using Lemma 3.3, a sequence $\langle \lambda_i : i \in \omega \rangle$ cofinal in $\lambda$, such that each $L_{\lambda_i} \prec L_\lambda$, and $U \cap L_{\lambda_i}, L_{\lambda_i} \in L_{\lambda_{i+1}}$. Let $j_i$ be the restriction of $j$ to $L_{\lambda_i}$. Each $j_i : L_{\lambda_i} \to L_{j(\lambda_i)}$ is an element of $L$ by remark 3.1 (2), since it has size $\kappa$ in $L_\lambda$. These observations motivate the construction below.

To show that $\kappa$ is 1-iterable in $L$, for every $A \subseteq \kappa$ in $L$, we need to construct in $L$ a weak $\kappa$-model $M$ containing $A$ and a 1-good $M$-ultrafilter on $\kappa$. Fix $A \subseteq \kappa$ in $L$. Define in $L$, the tree $T$ of finite sequences of the form

$$s = \langle h_0 : L_{\gamma_0} \to L_{\delta_0}, \ldots, h_n : L_{\gamma_n} \to L_{\delta_n} \rangle$$

ordered by extension and satisfying the properties:

(1) $A \in L_{\gamma_0} \models \mathrm{ZFC}^-$,
(2) $h_i : L_{\gamma_i} \to L_{\delta_i}$ is an elementary embedding with critical point $\kappa$,
(3) $\delta_i < j(\lambda)$.

Let $W_i$ be the $L_{\gamma_i}$-ultrafilter generated by $\kappa$ via $h_i$. Then:

(4) for $i < j \leq n$, we have $L_{\gamma_i}, W_i \in L_{\gamma_j}$, $L_{\gamma_i} \prec L_{\gamma_j}$, $L_{\delta_i} \prec L_{\delta_j}$, and $h_j$ extends $h_i$.

We view the sequences $s$ as better and better approximations to the embedding we are trying to build.

Consider the sequences

$$s_n = \langle j_0 : L_{\lambda_0} \to L_{j(\lambda_0)}, \ldots, j_n : L_{\lambda_n} \to L_{j(\lambda_n)} \rangle.$$

Clearly each $s_n$ is an element of $T$ and $\langle s_n : n \in \omega \rangle$ is a branch through $T$ in $V$. Hence the tree $T$ is ill-founded. By absoluteness, it follows that $T$ is ill-founded in $L$ as well. Let $\{h_i : L_{\gamma_i} \to L_{\delta_i} : i \in \omega\}$ be a branch of $T$ in $L$ and $W_i$ be the $L_{\gamma_i}$-ultrafilters as above. Let

$$h = \cup_{i \in \omega} h_i,\ L_\gamma = \cup_{i \in \omega} L_{\gamma_i},\ L_\delta = \cup_{i \in \omega} L_{\delta_i},\ \text{and } W = \cup_{i \in \omega} W_i.$$

It is clear that $h : L_\gamma \to L_\delta$ is an elementary embedding with critical point $\kappa$ and $W$ is a weakly amenable $L_\gamma$-ultrafilter generated by $\kappa$ via $h$. Since the ultrapower of $L_\gamma$ by $W$ is a factor embedding of $h$, it must be well-founded.

We have now found a weak $\kappa$-model $L_\gamma$ containing $A$ for which there exists a 1-good $L_\gamma$-ultrafilter on $\kappa$. This completes the proof that $\kappa$ is 1-iterable. □

The next lemma will allow us to modify the proof of Theorem 3.4 to show that 1-iterable cardinals are downward absolute to $L$.

**Lemma 3.5.** *If $\kappa$ is a 1-iterable cardinal, then every $A \subseteq \kappa$ is contained in a weak $\kappa$-model $M$ for which there exists a 1-iterable $M$-ultrafilter $U$ on $\kappa$ satisfying the conditions:*

1. *$M = \cup_{n\in\omega} M_n$,*
2. *$M_n, M_n \cap U \in M_{n+1}$,*
3. *for $i < j$, we have $M_i \prec M_j$,*
4. *$M \models$ "I am $H_{\kappa^+}$" (every set has transitive closure of size at most $\kappa$).*

*Proof.* Fix $A \subseteq \kappa$ and find a weak $\kappa$-model $M'$ containing $A$ with a 1-good $M'$-ultrafilter $U'$ on $\kappa$. Let $h : M' \to N'$ be the ultrapower embedding by $U'$. We can assume without loss of generality that $M' \models$ "I am $H_{\kappa^+}$" by taking $\{B \in M' \mid M' \models$ transitive closure of $B$ has size $\leq \kappa\}$ instead of $M'$ and restricting the embedding accordingly. Since $M' \models$ "I am $H_{\kappa^+}$" and $h$ is $\kappa$-powerset preserving, it follows that $M' = H^{N'}_{\kappa^+}$. In $N'$, let $M_0$ be a transitive elementary submodel of $H_{\kappa^+}$ of size $\kappa$ containing $A$. Since $U'$ is weakly amenable, it follows that $U_0 = M_0 \cap U'$ is an element of $H^{N'}_{\kappa^+}$. Let $M_1$ be a transitive elementary submodel of $H_{\kappa^+}$ of size $\kappa$ containing $M_0$ and $U_0$ and let $U_1 = M_1 \cap U'$. Again, $U_1$ is clearly an element of $H^{N'}_{\kappa^+}$. Inductively let $M_{n+1}$ be a transitive elementary submodel of $H_{\kappa^+}$ of size $\kappa$ containing $M_n$ and $U_n$ and $U_{n+1} = M_{n+1} \cap U'$. Let $M = \cup_{n\in\omega} M_n$ and $U = \cup_{n\in\omega} U_n$. Clearly $U$ is a weakly amenable $M$-ultrafilter and the ultrapower of $M$ by $U$ is well-founded as it embeds into $N$. □

**Theorem 3.6.** *If $\kappa$ is 1-iterable, then $\kappa$ is 1-iterable in $L$.*

*Proof.* Observe that if $0^\#$ exists, the theorem follows from Theorem 3.4 since all uncountable cardinals of $V$ are among the Silver indiscernibles. So suppose $0^\#$ does not exist. In $L$, fix $L_\xi$ of size $\kappa$. Choose a weak $\kappa$-model $M$ containing $L_\xi$ and $V_\kappa$ for which there exists a 1-iterable $M$-ultrafilter $U$ and let $j : M \to N$ be the ultrapower embedding. It is easy to see that $\kappa$ is weakly compact in $N$, and hence in $V^N_{j(\kappa)} \models$ ZFC. Since $V^N_{j(\kappa)}$ knows that $0^\#$ does not exist, it must satisfy that $(\kappa^+)^L = \kappa^+$. Restrict to $j : L_\alpha \to L_\beta$ where $\alpha$ and $\beta$ are the heights of $M$ and $N$ respectively. By the observation above, $(\kappa^+)^{L_\beta} = \alpha$ and hence the restriction is $\kappa$-powerset preserving. Note also, that by Lemma 3.5, we can assume that $\mathrm{cf}^V(\alpha) = \omega$. Therefore the embedding $j : L_\alpha \to L_\beta$ has exactly the same properties as the embedding with the Silver indiscernible as the critical point. So we can proceed as in the proof of Theorem 3.4 to construct a weak $\kappa$-model containing $L_\xi$ and a 1-iterable ultrafilter for it in $L$. □

The next lemma is a simple observation that will prove key to generalizing the arguments above for $\alpha$-iterable cardinals. Let us say that

$$\{j_{\xi\gamma} : M_\xi \to M_\gamma \mid \xi < \gamma < \alpha\}$$

is a *good commuting system of elementary embeddings of length $\alpha$* if:

(1) for all $\xi_0 < \xi_1 < \xi_2 < \alpha$, $j_{\xi_1\xi_2} \circ j_{\xi_0\xi_1} = j_{\xi_0\xi_2}$.

Let $\kappa_\xi$ be the critical point of $j_{\xi\xi+1}$ and let $U_\xi$ be the $M_\xi$-ultrafilter generated by $\kappa_\xi$ via $j_{\xi\xi+1}$, then:

(2) for all $\xi < \beta < \alpha$, if $A \in M_\xi$ and $A \subseteq U_\xi$, then $j_{\xi\beta}(A) \subseteq U_\beta$.

**Remark 3.7.** The directed system of embeddings resulting from the iterated ultrapowers construction is a good commuting system of elementary embeddings.

The next lemma shows that existence of good commuting systems of elementary embeddings of length $\alpha$ is basically equivalent to existence of $\alpha$-good ultafilters.

**Lemma 3.8.** *Suppose $\{j_{\xi\gamma} : M_\xi \to M_\gamma : \xi < \gamma < \alpha\}$ is a good commuting system of elementary embeddings of length $\alpha$. Suppose further that $m_0 = \cup_{i\in\omega} m_0^{(i)}$ is a transitive elementary submodel of $M_0$ such that $m_0^{(i)} \prec m_0^{(i+1)}$ and $U_0 \cap m_0^{(i)}$, $m_0^{(i)} \in m_0^{(i+1)}$. Then $u_0 = m_0 \cap U_0$ is an $\alpha$-good $m_0$-ultrafilter.*

*Proof.* Let $\{h_{\xi\gamma} : m_\xi \to m_\gamma : \xi < \gamma < \alpha\}$ be the not necessarily well-founded directed system of embeddings obtained by iterating $u_0$ and let $u_\xi$ be the $\xi^{\text{th}}$-iterate of $u_0$. Define $u_0^{(i)} = u_0 \cap m_0^{(i)}$ and $u_\xi^{(i)} = h_{0\xi}(u_0^{(i)})$. It is easy to see that $h_{\beta\xi}(u_\beta^{(i)}) = u_\xi^{(i)}$ and $u_\xi = \cup_{i\in\omega} u_\xi^{(i)}$. To show that each $m_\xi$ is well-founded, we will argue that we can define elementary embeddings $\pi_\xi : m_\xi \to M_\xi$. More specifically we will construct the following commutative diagram:

$$\begin{tikzcd}
M_0 \arrow[r, "j_{01}"] & M_1 \arrow[r, "j_{12}"] & M_2 \arrow[r, "j_{23}"] & \dots \arrow[r, "j_{\xi\xi+1}"] & M_{\xi+1} \arrow[r, "j_{\xi+1\xi+2}"] & \dots \\
m_0 \arrow[u, "\pi_0"] \arrow[r, "h_{01}"] & m_1 \arrow[u, "\pi_1"] \arrow[r, "h_{12}"] & m_2 \arrow[u, "\pi_2"] \arrow[r, "h_{23}"] & \dots \arrow[r, "h_{\xi\xi+1}"] & m_{\xi+1} \arrow[u, "\pi_{\xi+1}"] \arrow[r, "h_{\xi+1\xi+2}"] & \dots
\end{tikzcd}$$

where

(1) $\pi_{\xi+1}([f]_{u_\xi}) = j_{\xi\xi+1}(\pi_\xi(f))(\kappa_\xi)$,
(2) if $\lambda$ is a limit ordinal and $t$ is a thread in the direct limit $m_\lambda$ with domain $[\beta, \lambda)$, then $\pi_\lambda(t) = j_{\beta\lambda}(\pi_\beta(t(\beta)))$,
(3) $\pi_\xi(u_\xi^{(i)}) \subseteq U_\xi$ for all $i \in \omega$.

We will argue that the $\pi_\xi$ exist by induction on $\xi$. Let $\pi_0$ be the identity map. Suppose inductively that $\pi_\xi$ has the desired properties. Define $\pi_{\xi+1}$ as in (1) above. Since $\pi_\xi(u_\xi^{(i)}) \subseteq U_\xi$ by the inductive assumption, it follows that $\pi_{\xi+1}$ is a well-defined elementary embedding. The commutativity of the diagram is also clear. It remains to verify that $\pi_{\xi+1}(u_{\xi+1}^{(i)}) \subseteq U_{\xi+1}$. Recall that

$$u_{\xi+1}^{(i)} = h_{\xi\xi+1}(u_\xi^{(i)}) = [c_{u_\xi^{(i)}}]_{u_\xi}.$$

Let $\pi_\xi(u_\xi^{(i)}) = v$. Then by inductive assumption, $v \subseteq U_\xi$. Thus,

$$\pi_{\xi+1}(u_{\xi+1}^{(i)}) = j_{\xi\xi+1}(c_v)(\kappa_\xi) = c_{j_{\xi\xi+1}(v)}(\kappa_\xi) = j_{\xi\xi+1}(v).$$

By hypothesis, $j_{\xi\xi+1}(v) \subseteq U_{\xi+1}$. This completes the inductive step. The limit case also follows easily. □

Below, we give another useful example of a good commuting system of elementary embeddings.

**Lemma 3.9.** *Suppose $L$ (or $L_\rho$) is the Skolem closure of a collection of order indiscernibles $I$ and let $i_\xi$ be the $\xi^{th}$ element of $I$. Suppose $\delta$ is below o.t.$(I)$ and $\lambda$ is an ordinal such that for all $\xi$ below o.t.$(I)$ and all $\xi' < \lambda$, the sum $\xi + \xi'$ is below o.t.$(I)$. Then the system of embeddings $\{j_{\alpha\beta} \mid \alpha < \beta < \lambda\}$ defined by $j_{\alpha\beta}(i_\xi) = i_\xi$ for $\xi < \delta + \alpha$ and otherwise $j_{\alpha\beta}(i_\xi) = i_{\xi+\gamma}$ where $\alpha + \gamma = \beta$ is a good commuting system of elementary embeddings.*

Note that o.t.$(I)$ is allowed to be $Ord$. The proof is a straightforward application of indiscernibility. Thus, if $\kappa$ is a Silver indiscernible, then there is a good commuting system of elementary embeddings of length $Ord$ with the first embedding having critical point $\kappa$.

Now we can generalize Lemma 3.5 to the case of $\alpha$-iterable cardinals.

**Lemma 3.10.** *If $\kappa$ is an $\alpha$-iterable cardinal, then every $A \subseteq \kappa$ is contained in a weak $\kappa$-model $M$ for which there exists an $\alpha$-good $M$-ultrafilter $U$ on $\kappa$ satisfying conditions* (1)-(4) *of Lemma* 3.5*.*

*Proof.* Start with any weak $\kappa$-model $M'$ and an $\alpha$-good $M'$-ultrafilter $U'$ on $\kappa$. Use proof of Lemma 3.5 to find a transitive elementary submodel $M$ of $M'$ satisfying the requirements and use Lemma 3.8 to argue that $U = M \cap U'$ is $\alpha$-good. $\square$

We are now ready to show that if $0^\#$ exists, the Silver indiscernibles are $\alpha$-iterable in $L$ for all $\alpha < \omega_1^L$. It will follow using the same techniques that for $\alpha < \omega_1^L$, the $\alpha$-iterable cardinals are downward absolute to $L$.

**Theorem 3.11.** *If $0^\#$ exists, then the Silver indiscernibles are $\alpha$-iterable in $L$ for all $\alpha < \omega_1^L$.*

*Proof.* Fix a Silver indiscernible $\kappa_0$ and let $\lambda_0 = (\kappa_0^+)^L$. Let $U_0$ be an $\omega_1$-good $L_{\lambda_0}$-ultrafilter on $\kappa_0$ that exists by Lemma 3.8 combined with Lemma 3.9. Let

$$\{j_{\gamma\xi} : L_{\lambda_\gamma} \to L_{\lambda_\xi} : \gamma < \xi < \alpha\}$$

be the good commuting system of elementary embeddings obtained from the first $\alpha$-steps of the iteration. Let $\kappa_\xi$ be the critical point of $j_{\xi\xi+1}$ and let $U_\xi$ be the $\xi^{\text{th}}$-iterate of $U_0$. As before, we find a cofinal sequence $\langle \lambda_0^{(i)} : i \in \omega\rangle$ in $\lambda_0$ such that $L_{\lambda_0^{(i)}} \prec L_{\lambda_0}$ and $U_0 \cap L_{\lambda_0^{(i)}}, L_{\lambda_0^{(i)}} \in L_{\lambda_0^{(i+1)}}$. Let $U_0^{(i)} = U_0 \cap L_{\lambda_0^{(i)}}$, $L_{\lambda_\xi^{(i)}} = j_{0\xi}(L_{\lambda_0^{(i)}})$, and $U_\xi^{(i)} = j_{0\xi}(U_0^{(i)})$. Finally, let $j_{\gamma\xi}^{(i)}$ be the restriction of $j_{\gamma\xi}$ to $L_{\lambda_\gamma^{(i)}}$. Observe that each $j_{\gamma\xi}^{(i)}$ is an element of $L$ by remark 3.1 (2). As before, we will use these sequences to show that the tree we construct below is ill-founded.

To show that $\kappa_0$ is $\alpha$-iterable in $L$, for every $A \subseteq \kappa_0$ in $L$, we need to construct in $L$ a weak $\kappa_0$-model $M$ containing $A$ and an $\alpha$-good $M$-ultrafilter on $\kappa_0$. Fix $A \subseteq \kappa_0$ in $L$. Also, fix in $L$, a bijection $\rho : [\alpha]^2 \to \omega$.

Define in $L$, the tree $T$ of finite tuples of sequences $t = \langle s_0, \ldots, s_n\rangle$ where each $s_i$ is a sequence of length $n$ consisting of approximations to the elementary embedding from stage $\xi$ to stage $\beta$ where $\rho(\xi, \beta) = i$. That is

$$s_i = \langle h_{\xi\beta}^{(0)} : L_{\gamma_\xi^{(0)}} \to L_{\gamma_\beta^{(0)}}, \ldots, h_{\xi\beta}^{(n)} : L_{\gamma_\xi^{(n)}} \to L_{\gamma_\beta^{(n)}}\rangle$$

Note that if $t$ is an $m$-tuple, then we require all sequences in the tuple to have length $m$. We define $t \leq t'$ whenever the length of $t'$ is greater than or equal to the length of $t$ and the $i^{\text{th}}$ coordinate of $t'$ extends the $i^{\text{th}}$ coordinate of $t$. The sequences $s_i = \langle h_{\xi\beta}^{(j)} : L_{\gamma_\xi^{(j)}} \to L_{\gamma_\xi^{(j)}} \mid 0 \leq j \leq n\rangle$ are required to satisfy the properties:

(1) $A \in L_{\gamma_0^{(0)}} \models \mathrm{ZFC}^-$,
(2) $h^{(j)}_{\xi\beta} : L_{\gamma_\xi^{(j)}} \to L_{\gamma_\beta^{(j)}}$ are commuting elementary embeddings,
(3) $\gamma_\xi^{(j)} < \lambda_\alpha = \cup_{\xi<\alpha}\lambda_\xi$.

Let $\kappa_\xi$ be the critical points of $h^{(j)}_{\xi\xi+1}$ and let $W_\xi^{(j)}$ be the $L_{\gamma_\xi^{(j)}}$-ultrafilters generated by $\kappa_\xi$ via $h^{(j)}_{\xi\xi+1}$, then:

(4) for $j < k \leq n$, $L_{\gamma_\xi^{(j)}}, W_\xi^{(j)} \in L_{\gamma_\xi^{(k)}}$, $L_{\gamma_\xi^{(j)}} \prec L_{\gamma_\xi^{(k)}}$ and $h^{(k)}_{\xi\beta}$ extends $h^{(j)}_{\xi\beta}$,
(5) $h^{(j)}_{\xi\beta}(W_\xi^{(j)}) = W_\beta^{(k)}$.

As before, we argue using the iterated ultrapowers of $U_0$, that $T$ is ill-founded in $V$ and hence in $L$. Unioning up the branch in $L$, we obtain a good commuting system of elementary embeddings of length $\alpha$ and therefore an $\alpha$-good ultrafilter for the first model in the system by Lemma 3.8. □

**Theorem 3.12.** *For $\alpha < \omega_1^L$, if $\kappa$ is $\alpha$-iterable, then $\kappa$ is $\alpha$-iterable in $L$.*

*Proof.* Use the proof of Theorem 3.6 together with Lemma 3.8. □

## 4. The hierarchy of $\alpha$-iterable cardinals

In this section, we show using the techniques developed in the previous section, that for $\alpha \leq \omega_1$, the $\alpha$-iterable cardinals form a hierarchy of strength. We make some observations about the relationship between $\alpha$-iterable cardinals and $\alpha$-Erdős cardinals. We show that a 2-iterable cardinal is a limit of Schindler's *remarkable* cardinals, improving the upper bound on their consistency strength, and that a remarkable cardinal implies the existence of a countable transitive model of ZFC with a proper class of 1-iterable cardinals. Finally, we answer a question from [Git09] about whether 1-iterable cardinals imply the existence of $\kappa$-powerset preserving embeddings on weak $\kappa$-models satisfying full ZFC.

**Theorem 4.1.** *If $\kappa$ is an $\alpha$-iterable cardinal, then for $\xi < \alpha$, the cardinal $\kappa$ is a limit of $\xi$-iterable cardinals.*

*Proof.* Suppose $\kappa$ is an $\alpha$-iterable cardinal. Choose a weak $\kappa$-model $M_0$ containing $V_\kappa$ as an element for which there exists an $\alpha$-good $M_0$-ultrafilter, satisfying the conclusions of Lemma 3.10. Let $j_\xi : M_\xi \to M_{\xi+1}$ be the $\xi^{\text{th}}$ step of the iteration by $U_0$. Suppose, first, that $\alpha = \beta + 1$ is a successor ordinal. In this case, the iteration will have a final model namely $M_\alpha$. It suffices to argue that $\kappa$ is $\beta$-iterable in $M_\alpha$. To see this, suppose that $\kappa$ is $\beta$-iterable in $M_\alpha$, then $\kappa$ is $\beta$-iterable in $M_1$ as well. But then $M_1$ satisfies that there is a $\beta$-iterable cardinal below $j_0(\kappa)$ and hence, by elementarity, $M_0$ satisfies that $\kappa$ is a limit of $\beta$-iterable cardinals. But since $V_\kappa \in M_0$, the model must be correct about this assertion. Now we exactly follow the argument that Silver indiscernibles are $\beta$-iterable in $L$, with $M_\alpha$ in the place of $L$. Let $M_0 = \cup_{i\in\omega} M_0^{(i)}$ where the $M_0^{(i)}$ satisfy the conclusions of Lemma 3.10 and let $M_\xi^{(i)} = j_{0\xi}(M_0^{(i)})$ for $\xi \leq \alpha$. Observe that for $\xi < \gamma \leq \beta$, the restrictions $j^{(i)}_{\xi\gamma} : M_\xi^{(i)} \to M_\gamma^{(i)}$ are all elements of $M_\beta$ by remark 3.1 (2). Moreover, $M_\beta$ is a set in $M_\alpha$ and hence the ordinals of $M_\beta$ are bounded in $M_\alpha$ by some ordinal $\delta$. This suffices to run the same tree building argument. The bound $\delta$ is needed to insure that the tree is a set. Next, suppose, that $\alpha$ is a limit ordinal. Fix $\xi < \alpha$, then $\kappa$ is $\xi+1$-iterable. So by the inductive assumption, $\kappa$ is a limit of $\xi$-iterable cardinals. □

Next, we give some results on the relationship between $\alpha$-iterable cardinals and $\alpha$-Erdős cardinals for $\alpha \leq \omega_1$.

**Definition 4.2.** Suppose $\kappa$ is a regular cardinal and $\alpha$ is a limit ordinal. Then $\kappa$ is *$\alpha$-Erdős* if every structure of the form $\langle L_\kappa[A], A\rangle$ where $A \subseteq \kappa$ has a good set of indiscernibles of order type $\alpha$.[4]

Equivalently, $\kappa$ is $\alpha$-Erdős if it is least such that the partition relation $\kappa \to (\alpha)^{<\omega}$ holds.

In [WS10], Sharpe and Welch showed that:

**Theorem 4.3.** *An $\omega_1$-Erdős is a limit of $\omega_1$-iterable cardinals.*

Here we show that:

**Theorem 4.4.** *If $\kappa$ is a $\gamma$-Erdős cardinal for some $\gamma < \omega_1$ and $\delta < \gamma$ is an ordinal such that for all $\xi' < \gamma$ and $\xi < \delta$, the sum $\xi' + \xi < \gamma$, then there is a countable ordinal $\alpha$ and a real $r$ such that $L_\alpha[r]$ is a model of* ZFC *having a proper class of $\delta$-iterable cardinals.*

*Proof.* Suppose $\kappa$ is a $\gamma$-Erdős cardinal, then there is a set $I = \{i_\xi \mid \xi \in \gamma\}$ of good indiscernibles for $L_\kappa[r]$ where $r$ codes the fact that $\gamma$ is a countable ordinal. Let $L_\alpha[r]$ be the collapse of the Skolem closure of $I$ in $L_\kappa[r]$, then it is the Skolem closure of some collection $K = \{k_\xi \mid \xi \in \gamma\}$ of indiscernibles. The indiscernibles $k_\xi$ are unbounded in $\alpha$. Since each $L_{i_\xi}[r] \prec L_\kappa[r]$, we have $L_\beta[r] \prec L_\kappa[r]$ where $\beta$ is the sup of $I$. It follows that the Skolem closure of $I$ is contained in $L_\beta[r]$ and hence the $k_\xi$ are unbounded in $L_\alpha[r]$. By Lemma 3.9, for every indiscernible $k_\xi$, there is a good commuting system of elementary embeddings of length $\delta$ on $L_\alpha[r]$ with the first embedding having critical point $k_\xi$. Thus, by Lemma 3.8, there is a $\delta$-good ultrafilter for every $L_{(k_\xi^+)^{L_\alpha[r]}}$. Since $\alpha$ is countable, it is clear that $(k_\xi^+)^{L_\alpha[r]}$ has countable cofinality. Now we can use the argument in the proof of Theorem 3.11 to show that each $k_\xi$ is $\delta$-iterable in $L_\alpha[r]$. Notice that we cannot use these techniques to make the argument for $\delta = \gamma$ since the tree of embedding approximations must be a set in $L_\alpha[r]$. □

In particular, note that an $\omega$-Erdős cardinal implies for every $n \in \omega$, the consistency of the existence of a proper class of $n$-iterable cardinals.

**Remark 4.5.** $\gamma$-Erdős cardinals do not necessarily have any iterability since the least such cardinal need not be weakly compact.

In [Sch04], Schinder defined *remarkable* cardinals and showed that they are equiconsistent with the assumption that $L(\mathbb{R})$ cannot be modified by proper forcing. Schindler showed that an $\omega$-Erdős cardinal implies that there is a countable model with a remarkable cardinal. We show that if $\kappa$ is 2-iterable, then $\kappa$ is a limit of remarkable cardinals. By Theorem 4.3, this is an improved upper bound on the consistency strength of these cardinals.

[4]See Section 6 for a discussion of *good* sets of indiscernibles.

**Definition 4.6.** A cardinal $\kappa$ is *remarkable* if for each regular $\lambda > \kappa$, there exists a countable transitive $M$ and an elementary embedding $e : M \to H_\lambda$ with $\kappa \in ran(e)$ and also a countable transitive $N$ and an elementary embedding $\theta : M \to N$ such that:

(1) $cp(\theta) = e^{-1}(\kappa)$,
(2) $Ord^M$ is a regular cardinal in $N$,
(3) $M = H^N_{Ord^M}$,
(4) $\theta(e^{-1}(\kappa)) > Ord^M$.

We will need the following property of 2-iterable cardinals.

**Theorem 4.7.** *If $\kappa$ is a 2-iterable cardinal, then every $A \subseteq \kappa$ is contained in a weak $\kappa$-model $M \models$ ZFC for which there exists an embedding $j : M \to N$ such that $M = V^N_{j(\kappa)}$ and $M \prec N$.*

For proof, it suffices to observe that if $U$ is a 2-good ultrafilter for a weak $\kappa$-model $M$, then we get the following commutative diagram:

$$\begin{tikzcd}
M \arrow[r, "j_U"] \arrow[d, "j_U"'] \arrow[dr, "j_{U^2}"] & N = M/U \arrow[d, "h_U"] \\
N = M/U \arrow[r, "j_{j_U(U)}"'] & K = N/j_U(U)
\end{tikzcd}$$

where $j_U$ and $h_U$ are ultrapowers by $U$ and $j_{j_U(U)}$ is the ultrapower by $j_U(U)$. If $V_\kappa \in M$, the restriction of $h_U$ to $V^N_{j(\kappa)}$ has all the required properties. See [Git09] for details. The ultrafilter needs to be 2-good to ensure that the bottom arrow embedding has a well-founded target.

**Theorem 4.8.** *If $\kappa$ is 2-iterable, then $\kappa$ is a limit of remarkable cardinals.*

*Proof.* Suppose $\kappa$ is 2-iterable, then there is $j : M \to N$ as in Theorem 4.7 with $V_\kappa \in M$. It will suffice to argue that $\kappa$ is remarkable in $M$, since it will be remarkable in $N$ by elementarity, and hence a limit of remarkable cardinals. In $M$, fix a regular cardinal $\lambda > \kappa$. Continuing to work in $M$, find $X_0 \prec H_\lambda$ of size $\kappa$ such that $V_\kappa \cup \{\kappa\} \subseteq X_0$. By remark 3.1 (2), $j \restriction X_0 : X_0 \to j(X_0)$ is an element of $N$. In $N$, find $Y_0 \prec j(X_0)$ of size $\kappa$ such that $X_0 \cup j``X_0 \cup \{\lambda\} \subseteq Y_0$. Let $j_0 : X_0 \to Y_0$ such that $j_0(x) = j(x)$ for all $x \in X_0$, then $j_0$ is clearly elementary and an element of $N$. Let $Z_0 = Y_0 \cap H_\lambda$ (clearly $H^M_\lambda = H^N_\lambda$), then $Z_0 \in H_\lambda$. Back in $M$, find $X_1 \prec H_\lambda$ such that $Z_0 \subseteq X_1$ and in $N$, find $Y_1 \prec j(X_1)$ of size $\kappa$ and containing $X_1 \cup j``X_1$. Let $j_1 : X_1 \to Y_1$ such that $j_1(x) = j(x)$ for all $x \in X_1$, then as before $j_1$ is elementary and an element of $N$. Proceed inductively to define the the sequence $\langle j_n : X_n \to Y_n \mid n \in \omega\rangle$. The elements of the sequence are all in $N$, but the sequence itself need not be. As in the previous proofs, we will use a tree argument to find a sequence with similar properties in $N$ itself. The elements of the tree $T$ will be sequences $\langle h_0 : P_0 \to R_0, \ldots, h_n : P_n \to R_n\rangle$ ordered by extension and satisfying the properties:

(1) $h_i : P_i \to R_i$ is an elementary embedding with critical point $\kappa$,
(2) $V_\kappa \cup \{\kappa\} \subseteq P_0$, $P_i \in H_\lambda$, $P_i$ has size $\kappa$ in $H_\lambda$, $P_i \prec H_\lambda$, and $P_i \subseteq R_i$
(3) $R_i \in H_{j(\lambda)}$ and $R_i$ has size $\kappa$,
(4) for $i < j \leq n$, $R_i \cap H_\lambda \subseteq P_j$.

The sequence of embeddings we constructed above is a branch through $T$ and hence $T$ is ill-founded. Thus, $N$ has a branch of $T$. Let $h : P \to R$ be the embedding obtained from unioning up the branch. By our construction, $P \prec H_\lambda$, $P = H_\lambda^R$, and $P$ is an element of $M$. Collapse $R$ and use the collapse to define an elementary embedding of transitive structures $\bar{h} : \bar{P} \to \bar{R}$ where $\bar{R}$ is the collapse of $R$ and $\bar{P}$ is the collapse of $P$. Since $\bar{h}$, $\bar{P}$, and $\bar{R}$ all have transitive size $\kappa$, they are elements of $M$. Observe that $Ord^{\bar{P}} = \gamma$ is a regular cardinal in $\bar{R}$, the critical point of $\bar{h}$ is $\kappa$, and $\bar{h}(\kappa) > \gamma$. Finally, in $M$, take a countable elementary substructure of $\langle \bar{R}, \bar{P}, \bar{h}\rangle$ and collapse the structures to obtain an elementary embedding $i : m \to n$ of countable structures with critical point $\theta$. Let $e : m \to H_\lambda$ be the composition of the inverses of the collapse maps. The embeddings $i : m \to n$ and $e : m \to H_\lambda$ clearly satisfy properties (1)-(4) in the definition of remarkable cardinals. This completes the argument that $\kappa$ is remarkable in $M$. $\square$

If $\kappa$ is at least 2-iterable, then by Theorem 4.7, we can assume without loss of generality that we have embeddings on weak $\kappa$-models satisfying full ZFC. Gitman asked in [Git09] whether the same holds true for 1-iterable cardinals. We end this section, by answering the question in the negative and using the same techniques to pin the consistency strength of remarkable cardinals exactly between 1-iterable and 2-iterable cardinals.

**Theorem 4.9.** *If every $A \subseteq \kappa$ can be put into a weak $\kappa$-model $M \models$ "$\mathcal{P}(\kappa)$ exists" for which there exists a 1-good $M$-ultrafilter on $\kappa$, then $\kappa$ is a limit of 1-iterable cardinals.*

*Proof.* Fix a weak $\kappa$-model $M \models$ "$\mathcal{P}(\kappa)$ exists" containing $V_\kappa$ for which there is a 1-good $M$-ultrafilter $U$, and let $j : M \to N$ be the ultrapower embedding. Fix $A \subseteq \kappa$ in $N$ and find in $N$, a transitive $M_0 \prec H_{\kappa^+}$ of size $\kappa$ and containing $A$. As before, $j \restriction M_0 : M_0 \to j(M_0)$ is in $N$. Next, find a transitive $M_1 \prec H_{\kappa^+}$ of size $\kappa$ containing $M_0$ and $U \cap M_0$ and proceed inductively to define the sequence $\langle M_n \mid n \in \omega\rangle$ in this manner. Again, we construct a tree to obtain a sequence with similar properties in $N$ that will witness 1-iterability. That the tree can be defined in the first place is a consequence of the fact that $\mathcal{P}(j(\kappa))$ and hence $H_{j(\kappa)^+}$ exists in $N$ by elementarity. $\square$

**Corollary 4.10.** *If every $A \subseteq \kappa$ can be put into a weak $\kappa$-model $M \models$ ZFC for which there exists a 1-good $M$-ultrafilter on $\kappa$, then $\kappa$ is a limit of 1-iterable cardinals.*

**Theorem 4.11.** *If $\kappa$ is a remarkable cardinal, then there is a countable transitive model of* ZFC *with a proper class of 1-iterable cardinals.*

*Proof.* Fix a regular $\lambda > \kappa^+$ and let $e : M \to H_\lambda$, $\sigma : M \to N$ with critical point $e^{-1}(\kappa) = \delta$ be as in definition 4.6. Note that $\mathcal{P}(\delta)$ exists in $M$ since $\mathcal{P}(\kappa)$ exists in $H_\lambda$. Arguing exactly as in the proof of Theorem 4.9, we see that $\delta = e^{-1}(\kappa)$ is a limit of 1-iterable cardinals. Thus, $V_\delta^M$ is a countable transitive model of ZFC with a class of 1-iterable cardinals. $\square$

## 5. Ramsey-like cardinals and downward absoluteness to $K$

In this section, we show that the strongly Ramsey and super Ramsey cardinals introduced in [Git09] are downward absolute to the core model $K$.

**Definition 5.1.** A cardinal $\kappa$ is *strongly Ramsey* if every $A \subseteq \kappa$ is contained in a $\kappa$-model $M$ for which there exists a $\kappa$-powerset preserving elementary embedding $j : M \to N$.

**Definition 5.2.** A cardinal $\kappa$ is *super Ramsey* if every $A \subseteq \kappa$ is contained in a $\kappa$-model $M \prec H_{\kappa^+}$ for which there exists a $\kappa$-powerset preserving elementary embedding $j : M \to N$.

Strongly Ramsey cardinals are limits of *completely Ramsey* cardinals that top Feng's $\Pi_\alpha$-Ramsey hierarchy [Fen90]. They are Ramsey, but not necessarily completely Ramsey. They were introduced with the motivation of using them for indestructibility arguments involving Ramsey cardinals. Such an application is made in Section 6. Super Ramsey cardinals are limits of strongly Ramsey cardinals and have the advantage that the embedding is on a $\kappa$-model that is stationarily correct. Note that we can restate the definition of strongly Ramsey and super Ramsey cardinals in terms of the existence of weakly amenable $M$-ultrafilters. Since we require the embedding to be on a $\kappa$-model, such an ultrafilter is automatically countably complete and therefore has a well-founded ultrapower.

As a representative *core model* $K$ here we take that constructed using extender sequences which are *non-overlapping* (see [Zem02]). In such a model a strong cardinal may exist but not a sharp for such. The argument does not depend on any particular fine structural considerations, simply the definability of $K$ up to $\kappa^+$ in any $H_{\kappa^+}$ with applications of the Weak Covering Lemma (*cf.* [Zem02]).

**Proposition 5.3.** *If $\kappa$ is strongly Ramsey, then $\kappa$ is strongly Ramsey in $K$.*

*Proof.* Let $\kappa$ be strongly Ramsey and fix $A \subseteq \kappa$ in $K$. Choose a $\kappa$-model $M$ containing $A$ such that $M \models A \in K$ for which there exists a weakly amenable $M$-ultrafilter $U$ on $\kappa$. To see that we can choose such $M$, note that $A \in P = \langle J_\alpha^{E^K}, \in, E^K \rangle$ for some $\alpha < \kappa^+$ where $E^K$ is the extender sequence from which $K$ is constructed. We may assume that a code for $P$ is definable over $V$ as a subset of $\kappa$. Hence we may assume that the $M$ witnessing strong Ramseyness has this code, and so $P$, as an element. Note that with $P \in M$, $K_\kappa$ is an initial segment of $K^M$. Finally observe that $V_\kappa \in M$ since $M$ is a $\kappa$-model. Let $\bar{\kappa} = (\kappa^+)^{K_M}$ and set $\bar{K} = K^M_{\bar{\kappa}}$. The possibility that $\bar{\kappa} = Ord^M$ is allowed.

Note that $P \in \bar{K}$ and moreover a standard comparison argument shows that $\bar{K}$ is an initial segment of $K$. Consider the structure $N = \langle \bar{K}, \in, W \rangle$ where $W = U \cap \bar{K}$, and observe that $W$ is a weakly amenable $\bar{K}$-ultrafilter. Note that $\mathrm{cf}(\bar{\kappa}) = \kappa$. If $\bar{\kappa} = (\kappa^+)^M$, this follows since $M$ is a $\kappa$-model. Otherwise, consider the inner model $W_M = \cup_{\alpha \in Ord} H^{M_\alpha}_{\kappa_\alpha}$ obtained by iterating the ultrafilter $U$ out through the ordinals, in which $\bar{\kappa}$ remains the $K$-successor of $\kappa$, and apply the Weak Covering Lemma to $\bar{\kappa}$. Hence $N$ is a premouse iterable by the ultrafilter $W$. This allows us to coiterate $N$ with $K$. We note that for no $\mu < \kappa$ do we have $o^K(\mu) \geq \kappa$, , that is $\kappa$ is not overlapped by any extender on a critical point $\mu$ below $\kappa$ since otherwise the ultrafilter $W$ would generate the sharp for an inner model with a strong cardinal and we are only considering $K$ build using non-overlapping extenders. Hence if $K$ were to move in this coiteration, either $\bar{\kappa} = (\kappa^+)^K$ and $\kappa$ is measurable in $K$ (and hence already strongly Ramsey) or else $K$ is first truncated to some $N' \in K$, $N' = \langle \bar{K}, \in, F\rangle$ with a weakly amenable $\bar{K}$-ultrafilter $F$. The next paragraph shows that $N'$ witnesses the strong Ramsey property for $A$.

It remains to show that $\bar{K}^{<\kappa} \subseteq \bar{K}$ in $K$. Fix $\alpha < \kappa$ and $f : \alpha \to \bar{K}$ in $K$. Without loss of generality we shall assume that $f$ is 1-1. Since $M$ is a $\kappa$-model, we have $f \in M$. Suppose $f(\gamma) \in \bar{K}_{\delta(\gamma)}$ for some $\delta(\gamma) < \bar{\kappa}$. Let $\delta = \sup_{\gamma<\alpha} \delta(\gamma)$. Then $\delta < \bar{\kappa}$. Let $G \in \bar{K}$ be such that $G : \kappa \to \bar{K}_\delta$ is a bijection. Then if $C = G^{-1}$ "ran$(f)$, then $C$ is a bounded subset of $\kappa$ with $C \in K$. However $C \in V_\kappa \to C \in K^M$. As both $f, G$ are (1-1) there is a permutation $\pi : \alpha \longrightarrow C$ with $f = G \circ \pi$. Again $\pi \in V_\kappa \cap K \cap M$. Hence $f \in \bar{K}$. □

**Proposition 5.4.** *If $\kappa$ is super Ramsey, then $\kappa$ is super Ramsey in $K$.*

*Proof.* Note that if $M \prec H_{\kappa^+}$, then $K^M \prec (K)^{H_{\kappa^+}} = K_{\kappa^+}$. Let $\bar{\kappa} = (\kappa^+)^K$. Then also $K^M \cap K_{\bar{\kappa}} \prec K_{\bar{\kappa}}$. Let $\bar{K} = K^M \cap K_{\bar{\kappa}}$. Now argue as in the last proposition using $N = \langle \bar{K}, \in, U \cap \bar{K}\rangle$ where $U$ is the filter weakly amenable to $M$. □

## 6. Virtually Ramsey cardinals

In [WS10], Sharpe and Welch defined a new large cardinal notion, the *virtually Ramsey* cardinal. Virtually Ramsey cardinals are defined by an apparently weaker statement about the existence of good indiscernibles than Ramsey cardinals. The definition was motivated by the conditions needed to get an upper bound on the consistency strength of the Intermediate Chang's Conjecture. In this section, we separate the notions of Ramsey and virtually Ramsey cardinals using an old forcing argument of Kunen's showing how to destroy and then resurrect a weakly compact cardinal [Kun78].

**Definition 6.1.** Suppose $\kappa$ is a cardinal and $A \subseteq \kappa$. Then $I \subseteq \kappa$ is a *good set of indiscernibles* for $\langle L_\kappa[A], A\rangle$ if for all $\gamma \in I$:

(1) $\langle L_\gamma[A \cap \gamma], A \cap \gamma\rangle \prec \langle L_\kappa[A], A\rangle$.
(2) $I \setminus \gamma$ is a set of indiscernibles for $\langle L_\kappa[A], A, \xi\rangle_{\xi\in\gamma}$.

**Remark 6.2.** If for every $A \subseteq \kappa$, there is $\gamma < \kappa$ such $\langle L_\gamma[A \cap \gamma], A \cap \gamma\rangle \prec \langle L_\kappa[A], A\rangle$, then it is easy to see that $\kappa$ must be inaccessible. Also, if $I$ is a set of good indiscernibles for $\langle L_\kappa[A], A\rangle$ and $|I| \geq 3$, then by clause (2), every $\gamma \in I$ is inaccessible in $\langle L_\kappa[A], A\rangle$.

**Theorem 6.3.** *A cardinal $\kappa$ is Ramsey if and only if for every $A \subseteq \kappa$, the structure $\langle L_\kappa[A], A\rangle$ has a good set of indiscernibles of size $\kappa$.*

See [DL89] for details on good sets of indiscernibles and proof of above theorem. In general, for $A \subseteq \kappa$, let $\mathscr{I}_A = \{\alpha \in \kappa \mid$ there is an unbounded set of good indiscernibles $I_\alpha \subseteq \alpha$ for $\langle L_\kappa[A], A\rangle\}$.

**Definition 6.4.** A cardinal $\kappa$ is *virtually Ramsey* if for every $A \subseteq \kappa$, the set $\mathscr{I}_A$ contains a club.[5]

[5]The original definition in [WS10] required that $\mathscr{I}_A$ contain only an $\omega_1$-club.

There is no obvious reason to suppose that the good sets of indiscernibles below each of the ordinals in $\mathscr{I}_A$ can be glued together into a good set of indiscernibles of size $\kappa$, suggesting that virtually Ramsey cardinals are not necessarily Ramsey. First, we make some easy observations about virtually Ramsey cardinals.

**Proposition 6.5.** *Ramsey cardinals are virtually Ramsey.*

*Proof.* Suppose $\kappa$ is a Ramsey cardinal. If $A \subseteq \kappa$, then there is a good set indiscernibles $I$ of size $\kappa$ for the structure $\langle L_\kappa[A], A\rangle$. Clearly the club of all limit points of $I$ is contained in $\mathscr{I}_A$. This verifies that $\kappa$ is virtually Ramsey. $\square$

The next proposition confirms that being a virtually Ramsey cardinal is a large cardinal notion.

**Proposition 6.6.** *Virtually Ramsey cardinals are Mahlo.*

*Proof.* Suppose $\kappa$ is virtually Ramsey. By remark 6.2, $\kappa$ is inaccessible. To see that $\kappa$ is Mahlo, let $A \subseteq \kappa$ code $H_\kappa$ and $C \subseteq \kappa$ be a club. If $I$ is any good set of indiscernibles for $L_\kappa[A, C]$ and $\gamma \in I$, then $\gamma \in C$ by (1) of 6.1. By remark 6.2, $L_\kappa[A, C]$ thinks that $\gamma$ is inaccessible but it is correct about this since it contains all of $H_\kappa$. $\square$

Next, we give a sufficient condition needed to glue the good sets of indiscernibles below the ordinals in $\mathscr{I}_A$ into a good set of indiscernibles of size $\kappa$.

**Proposition 6.7.** *If a cardinal is virtually Ramsey and weakly compact, then it is Ramsey.*

*Proof.* Suppose $\kappa$ is virtually Ramsey and weakly compact. We will argue that we can glue together the good sets of indiscernibles coming from the different ordinals of the club contained in $\mathscr{I}_A$ into a good set of indiscernibles of size $\kappa$. Fix $A \subseteq \kappa$ and let $C$ be a club contained in $\mathscr{I}_A$. Fix any weak $\kappa$-model $M$ containing $A$, $C$ and $V_\kappa$ as elements. By weak compactness, there exists an embedding $j : M \to N$ with critical point $\kappa$. Observe that $M \models C \subseteq \mathscr{I}^M_A$, where $\mathscr{I}^M_A$ is the set $\mathscr{I}_A$ defined from the perspective of $M$. By elementarity $N \models j(C) \subseteq \mathscr{I}^N_{j(A)}$. Since $\kappa \in j(C)$, it follows that there is a good set of indiscernibles for $\langle L_{j(\kappa)}[j(A)], j(A)\rangle$ below $\kappa$. But since $j(A) \cap \kappa = A$, it is easy to see that this is a good set of indiscernibles for $\langle L_\kappa[A], A\rangle$ as well. This completes the proof that $\kappa$ is Ramsey. $\square$

Our strategy to separate virtually Ramsey and Ramsey cardinals will be to start with a Ramsey cardinal and force to destroy its weak compactness while preserving virtual Ramseyness. Although ideally we would like to start with a Ramsey cardinal, we will have to start with a strongly Ramsey cardinal instead. The reason being that strongly Ramsey cardinals have embeddings on sets with $< \kappa$-closure that is required for indestructibility techniques. The argument below was worked out jointly with Joel David Hamkins and we would like to thank him for his contribution.

The forcing we use is Kunen's well-known forcing from [Kun78] to destroy and then resurrect weak compactness. The next lemma is a key observation in the argument.

**Lemma 6.8.** *If $\mathbb{P}$ is a $< \kappa$-distributive, stationary preserving forcing, $G \subseteq \mathbb{P}$ is $V$-generic and $\kappa$ is virtually Ramsey in $V[G]$, then $\kappa$ was already virtually Ramsey in $V$.*

*Proof.* Fix $A \subseteq \kappa$. Since $\mathbb{P}$ is $<\kappa$-distributive, it cannot add any new good sets of indiscernibles to ordinals $\alpha < \kappa$. It follows that $\mathscr{I}_A = \mathscr{I}_A^{V[G]}$. If $\mathscr{I}_A$ does not contain a club in $V$, then the complement $\overline{\mathscr{I}}_A$ is stationary in $V$. Since $\mathbb{P}$ is stationary preserving, $\overline{\mathscr{I}}_A$ remains stationary in $V[G]$. This is clearly a contradiction since $\kappa$ is virtually Ramsey in $V[G]$ and hence $\mathscr{I}_A$ contains a club. □

First, we define a forcing $\mathbb{Q}$ to add a Souslin tree $T$ together with a group of automorphisms $\mathscr{G}$ that acts *transitively* on $T$. A group of automorphisms $\mathscr{G}$ of a tree $T$ is said to act transitively if for every $a$ and $b$ on the same level of $T$, there is $\pi \in \mathscr{G}$ with $\pi(a) = b$. The elements of $\mathbb{Q}$ will be pairs $(t, f)$ where $t$ is a *normal* $\alpha+1$-tree for some $\alpha < \kappa$ such that $Aut(t)$ acts transitively and $f : \lambda \underset{\text{onto}}{\overset{1-1}{\to}} Aut(t)$ is some enumeration of $Aut(t)$. We have $(t_1, f_1) \leq (t_0, f_0)$ when

(1) $t_1$ end-extends $t_0$,
(2) for all $\xi \in Dom(f_0)$, $f_1(\xi)$ extends $f_0(\xi)$.

The strategy will be to force with $\mathbb{Q}$ to add a Souslin tree $T$ thereby destroying the strong Ramseyness of $\kappa$ and then to force with $T$ itself to resurrect it. The argument that the second forcing resurrects the strong Ramseyness of $\kappa$ will rely on the fact that the combined forcing $\mathbb{Q}$ followed by $T$ has a dense subset that is $<\kappa$-closed. It is to obtain this result that the usual forcing to add a Souslin tree needs to be augmented with the automorphism groups.

To show that the generic $\kappa$-tree $T$ added by $\mathbb{Q}$ is Souslin, we need to argue that every maximal antichain of $T$ is bounded. In the usual forcing to add a Souslin tree, the conditions are normal $\alpha+1$-trees and the argument is made by proving the Sealing Lemma. The Sealing Lemma states that if a condition forces that $\dot{A}$ is a name for a maximal antichain, then there is a stronger condition forcing that it is bounded. The argument for the Sealing Lemma goes as follows:
Suppose $t_0 \Vdash \dot{A}$ is a maximal antichain of $\dot{T}$. Choose $t_1 \leq t_0$ such that for every $s \in t_0$, there is $a_s \in t_1$ compatible with $s$ and $t_1 \Vdash a_s \in \dot{A}$. Build a sequence $\cdots \leq t_n \leq \cdots \leq t_1 \leq t_0$ such that for every $s \in t_n$, there is $a_s \in t_{n+1}$ compatible with $s$ and $t_{n+1} \Vdash a_s \in \dot{A}$. Let $t$ be the union of $t_n$ and build the top level of $t$ by adding a branch through every pair $s$ and $a_s$. Since every new branch passes through an element of $\dot{A}$, this *seals* the antichain. We will carry out a similar argument with the forcing $\mathbb{Q}$, but it will be complicated by the fact that whenever we add a node on top of a branch $B$, we need to add nodes on top of branches $f(\xi)$"$B$. While $B$ passes through an element of $\dot{A}$, there is no reason why $f(\xi)$"$B$ should. In fact, since the automorphism groups act transitively, it will suffice to add a single carefully chosen branch to the limit tree of the conditions and take the limit level to be all the images of the branch under the automorphism group on the second coordinate. Thus, we need to build our sequence of conditions such that the limit of the trees on the sequence has a branch all of whose images under the automorphism group go through elements of the antichain.

**Lemma 6.9** (Sealing Lemma)**.** *Suppose $p$ is a condition in $\mathbb{Q}$, $\dot{T}$ is the canonical $\mathbb{Q}$-name for the generic $\kappa$-tree added by $\mathbb{Q}$, and $p \Vdash \dot{A}$ is a maximal antichain of $\dot{T}$. Then there is $q \leq p$ forcing that $\dot{A}$ is bounded.*

*Proof.* Fix $p \Vdash \dot{A}$ is a maximal antichain of $\dot{T}$. Let $p = (t_0, f_0)$ with $t_0$ of height $\alpha+1$ and $f_0 : \lambda_0 \underset{\text{onto}}{\overset{1-1}{\to}} Aut(t_0)$. Choose some $M \prec H_{\kappa^+}$ of size $<\kappa$ containing $\mathbb{Q}$, $p$, and $\dot{A}$

with the additional property that $Ord^M \cap \kappa = \beta$ is an initial segment of $\kappa$. We will work inside $M$ to build a condition $(t, f)$ with $t$ of height $\beta+1$ strengthening $(t_0, f_0)$ and sealing $\dot{A}$. We will need a bookkeeping function $\varphi : \kappa \underset{\text{onto}}{\to} \kappa$ with the property that every $\xi$ appears in the range cofinally often. Notice that by elementarity, $M$ contains some such function $\varphi$. Working *entirely inside* $M$, we carry out the following construction for $\kappa$ many steps. By going to a stronger condition, we can assume without loss of generality that there is $a \in t_0$ such that $(t_0, f_0) \Vdash a \in \dot{A}$. Let $B_0$ be any branch through $a$ in $t_0$. Let $a_0$ be the top node of $B_0$. The node $a_0$ begins the branch we are trying to construct. Let $(t_1, f_1)$ be a condition in $\mathbb{Q}$ strengthening $(t_0, f_0)$ and having the property that for every $s \in t_0$, there is $a_s \in t_1$ compatible with $s$ such that $(t_1, f_1) \Vdash a_s \in \dot{A}$. Consult the bookkeeping function $\varphi(1) = \gamma$. This will determine how $a_0$ gets extended. If $\gamma \geq \lambda_0$, let $a_1$ be the node on the top level of $t_1$ extending $a_0$. Otherwise, consider $f_0(\gamma)$ and $f_0(\gamma)(a_0) = s$. Let $s'$ be on the top level of $t_1$ above $s$ and $a_s$. Finally, let $f_1(\gamma)^{-1}(s') = a_1$. This has the effect that no matter how we extend $f_0(\gamma)$, the image under it of the branch we are building will pass through $\dot{A}$. At successor stages $\sigma + 1$, we will extend the condition $(t_\sigma, f_\sigma)$ to a condition $(t_{\sigma+1}, f_{\sigma+1})$ having the property that for every $s \in t_\sigma$, there is $a_s \in t_{\sigma+1}$ compatible with $s$ such that $(t_{\sigma+1}, f_{\sigma+1}) \Vdash a_s \in \dot{A}$. Next, we will consult the bookkeeping function $\varphi(\sigma) = \gamma$ and let it decide as above how $a_\sigma$ gets chosen. At limit stages $\lambda$, we will let $t_\lambda$ be the union of $t_\xi$ for $\xi < \lambda$ and $f_\lambda$ be the coordinate-wise union of $f_\xi$. Now use the branch through $a_\xi$ to define a limit level for $t_\lambda$, thereby extending to $(t_{\lambda+1}, f_{\lambda+1})$. From the perspective of $M$, we are carrying out this construction for $\kappa$ many steps, but really we are only carrying it out for $\beta$ many steps. In $V$, we build $(t, f)$ by unioning the sequence and adding a limit level using the branch of the $a_\xi$. It should be clear that $(t, f)$ forces that $\dot{A}$ is bounded. $\square$

**Corollary 6.10.** *The generic $\kappa$-tree added by $\mathbb{Q}$ is Souslin.*

In the generic extension by $\mathbb{Q}$, the Souslin tree $T$ it adds can be viewed as a poset. Next, we will argue that forcing with $\mathbb{Q} * \dot{T}$ is forcing equivalent $Add(\kappa, 1)$, where $Add(\kappa, 1)$ is the forcing to add a Cohen subset to $\kappa$. Since every $< \kappa$-closed poset of size $\kappa$ is forcing equivalent to $Add(\kappa, 1)$, it suffices argue that $\mathbb{Q} * \dot{T}$ has a dense subset that is $< \kappa$-closed.

**Lemma 6.11.** *The forcing $\mathbb{Q} * \dot{T}$ has a dense subset that is $< \kappa$-closed.*

*Proof.* Conditions in $\mathbb{Q} * \dot{T}$ are triples $(t, f, \dot{a})$ where $t$ is an $\alpha + 1$-tree, $f$ is an enumeration of the automorphism group of $t$, and $\dot{a}$ is a name for an element of $\dot{T}$. We will argue that conditions of the form $(t, f, a)$ where $a$ is on the top level of $t$ form a dense $< \kappa$-closed subset of $\mathbb{Q} * \dot{T}$. Start with any condition $(t_0, f_0, \dot{b}_0)$ and strengthen $(t_0, f_0)$ to a condition $(t_1, f_1)$ deciding that $\dot{b}$ is $b \in t_1$. Now we have $(t_0, f_0, \dot{b}) \geq (t_1, f_1, b) \geq (t_1, f_1, a)$ where $a$ is above $b$ on the top level of $t_1$. Thus the subset is dense. Suppose $\gamma < \kappa$ and we have a descending $\gamma$-sequence $(t_0, f_0, a_0) \geq (t_1, f_1, a_1) \geq \ldots \geq (t_\xi, f_\xi, a_\xi) \geq \ldots$. To find a condition that is above the sequence, we take unions of the first two coordinates and make the limit level of the tree in the first coordinate consist of images of the branch through $\langle a_\xi \mid \xi < \gamma \rangle$ under the automorphisms in the second coordinate. $\square$

Let $\mathbb{P}_\kappa$ be the Easton support iteration which adds a Cohen subset to every inaccessible cardinal below $\kappa$. We will force with the iteration $\mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{T}$. This is equivalent to forcing with $\mathbb{P}_\kappa * Add(\kappa, 1)$. The forcing argument will rely crucially on the following standard theorem about preservation of strong Ramsey cardinals after forcing.

**Theorem 6.12.** *If $\kappa$ is strongly Ramsey in $V$, then it remains strongly Ramsey after forcing with $\mathbb{P}_\kappa * Add(\kappa, 1)$.*

The proof uses standard techniques for lifting embeddings and will appear in [GJ10].

Now we have all the machinery necessary to produce a model where $\kappa$ is virtually Ramsey but not weakly compact.

**Theorem 6.13.** *If $\kappa$ is a strongly Ramsey cardinal, then there is a forcing extension, in which $\kappa$ is virtually Ramsey, but not weakly compact.*

*Proof.* Let $\kappa$ be a strongly Ramsey cardinal and $G * T * B \subseteq \mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{T}$ be $V$-generic. Since $\mathbb{Q}$ adds a Souslin tree, $\kappa$ is not weakly compact in the intermediate extension $V[G][T]$. But by Theorem 6.12, the strong Ramseyness of $\kappa$ is resurrected in $V[G][T][B]$. Recall that forcing with a Souslin tree is $< \kappa$-distributive. The Souslin tree forcing is also $\kappa$-cc and hence stationary preserving. So by Lemma 6.8, we conclude that $\kappa$ remains virtually Ramsey in $V[G][T]$. Thus, in $V[G][T]$, the cardinal $\kappa$ is virtually Ramsey but not weakly compact, and hence not Ramsey. $\square$

We showed starting from a strongly Ramsey cardinal that it is possible to have virtually Ramsey cardinals that are not Ramsey, thus separating the two notions. The following questions are still open.

**Question 6.14.** Can we separate Ramsey and virtually Ramsey cardinals starting with just a Ramsey cardinal?

**Question 6.15.** Are virtually Ramsey cardinals strictly weaker than Ramsey cardinals?

**Question 6.16.** Are virtually Ramsey cardinals downward absolute to $K$?

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