

A MITCHELL-LIKE ORDER FOR RAMSEY AND RAMSEY-LIKE CARDINALS

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ABSTRACT. Smallish large cardinals κ are often characterized by the existence of a collection of filters on κ , each of which is an ultrafilter on the subsets of κ of some transitive ZFC^- -model of size κ . We introduce a Mitchell-like order for Ramsey and Ramsey-like cardinals, ordering such collections of small filters. We show that the Mitchell-like order and the resulting notion of rank have all the desirable properties of the Mitchell order on normal measures on a measurable cardinal. The Mitchell-like order behaves robustly with respect to forcing constructions. We show that extensions with cover and approximation properties cannot increase the rank of a Ramsey or Ramsey-like cardinal. We use the results about extensions with cover and approximation properties together with recently developed techniques about soft killing of large-cardinal degrees by forcing to softly kill the ranks of Ramsey and Ramsey-like cardinals.

1. INTRODUCTION

Mitchell introduced the *Mitchell order* on normal measures on a measurable cardinal κ in [Mit74], where he defined that $U \triangleleft W$ for two normal measures U and W on κ whenever $U \in \text{Ult}(V, W)$, the ultrapower of the universe V by W . Since \triangleleft is easily seen to be well-founded, we can define the ordinal rank $o(U)$ of a normal measure and define $o(\kappa)$, the Mitchell rank of κ , to be the supremum of $o(U)$ over all normal measures U on κ . The Mitchell rank of κ tells us to what extent measurability is reflected below κ . Mitchell used the Mitchell order to study coherent sequences of normal measures, which allowed him to generalize Kunen's $L[U]$ construction to canonical inner models with many measures (cf. [Mit74]). The Mitchell rank of a measurable cardinal has also proved instrumental in calibrating consistency strength of set theoretic assertions. Gitik showed, for instance, that the consistency strength of a measurable cardinal at which the GCH fails is a measurable cardinal κ with $o(\kappa) = \kappa^{++}$ (cf. [Git93]). The notion of Mitchell order generalizes to extenders, where it has played a role in constructions of core models.

In this article, we introduce a Mitchell-like order for Ramsey and Ramsey-like cardinals. Although we tend to associate smaller large cardinals κ with combinatorial definitions, many of them have characterizations in terms of existence of elementary embeddings. The domains of these embeddings are *weak κ -models*, transitive models of ZFC^- of size κ and height above κ , or some stronger version of these such as *κ -models*, which are additionally closed under $<\kappa$ -sequences. Usually, the embeddings are ultrapower or extender embeddings by mini-measures or mini-extendors that apply only to the κ -sized domain of the embedding. If M is a transitive model of ZFC^- and κ is a cardinal in M , then we call $U \subseteq \mathcal{P}(\kappa)^M$ an

M -ultrafilter¹ if it is an ultrafilter on $\mathcal{P}(\kappa)^M$ that is normal for sequences in M . In most interesting cases, an M -ultrafilter is external to M , but we can still form the ultrapower by using functions on κ that are elements of M . A prototypical characterization of a smaller large cardinal κ states that every $A \subseteq \kappa$ is an element of a weak κ -model M (with additional requirements) for which there is an M -ultrafilter on κ (with additional requirements). The additional requirements on M and the M -ultrafilter are dictated by the large cardinal property. The simplest such characterization belongs to weakly compact cardinals, where there is the minimal requirement on the M -ultrafilter, namely that the ultrapower of M is well-founded.

Given a large-cardinal property \mathcal{P} with an embedding characterization as discussed above (such as weak compactness, Ramseyness, etc.), let us say that an M -ultrafilter is a \mathcal{P} -measure if it, together with M , witnesses \mathcal{P} and that a \mathcal{P} -measure is A -good for some $A \subseteq \kappa$ if $A \in M$.² To avoid having to specify which model M we associate to a given \mathcal{P} -measure U , we will always associate it with a unique minimal model M_U , namely the H_{κ^+} of any such model M . Let us say that a collection \mathcal{U} of \mathcal{P} -measures is a *witness* for \mathcal{P} if for every $A \subseteq \kappa$, it contains some A -good \mathcal{P} -measure. So while a normal measure on κ witnesses the measurability of κ , a witness collection of \mathcal{P} -measures is precisely what witnesses \mathcal{P} for one of these smaller large cardinals. This suggests that a reasonable Mitchell-like order should not be comparing the tiny \mathcal{P} -measures, but rather witness collections of \mathcal{P} -measures in a way that ensures that the corresponding rank $o_{\mathcal{P}}(\kappa)$ of κ measures the extent to which \mathcal{P} is reflected below κ . We will call this order the M-order in honor of Mitchell.

Definition 1.1 (M-order). Suppose that κ has a large-cardinal property \mathcal{P} with an embedding characterization. Given two witness collections \mathcal{U} and \mathcal{W} of \mathcal{P} -measures, we define that $\mathcal{U} \triangleleft \mathcal{W}$ if

- (1) For every $W \in \mathcal{W}$ and $A \subseteq \kappa$ in the ultrapower N_W of M_W by W , there is an A -good $U \in \mathcal{U} \cap N_W$ such that N_W agrees that U is an A -good \mathcal{P} -measure on κ .
- (2) $\mathcal{U} \subseteq \bigcup_{W \in \mathcal{W}} N_W$.

The key part of the definition is clause (1). It states that the elements of \mathcal{U} witness that κ retains the property \mathcal{P} in the ultrapowers by the elements of \mathcal{W} . It is tempting to say that \mathcal{U} itself should witness \mathcal{P} in those ultrapowers, but note that \mathcal{U} is too large to be an element of a weak κ -model. Clause (2) ensures that the collections of under consideration do not contain superfluous \mathcal{P} -measures.

Mitchell proved that Ramsey cardinals have an embedding characterization and Gitman used generalizations of it to define the Ramsey-like cardinals: α -iterable, strongly Ramsey, and super Ramsey cardinals (cf. [Mit79] and [Git11]). Thus, a Ramsey measure U is a weakly amenable ω_1 -intersecting M_U -ultrafilter, a strongly Ramsey measure U is a weakly amenable M_U -ultrafilter, where M_U is a κ -model, and a super Ramsey measure is a weakly amenable M_U -ultrafilter where M_U is a κ -model elementary in H_{κ^+} .

We will show that the M-order and the corresponding notion of M-rank share all the desirable features of the Mitchell order on normal measures on a measurable

¹The notation $\mathcal{P}(\kappa)^M$ is meant to denote $\mathcal{P}(\kappa) \cap M$, whether or not this is actually an element of M .

²For technical reasons we also require that $V_{\kappa} \in M$. Note that if M is a κ -model, then $V_{\kappa} \in M$ follows.

cardinal. For example, the order is transitive and well-founded. Note that since an ultrapower of a weak κ -model has size at most κ , the M-rank of a large cardinal κ can be at most κ^+ , in contrast with the upper bound of $(2^\kappa)^+$ in the case of the usual Mitchell rank for a measurable cardinal.

Theorem 1.2. *Suppose \mathcal{U} is a witness collection of \mathcal{P} -measures, where \mathcal{P} is Ramsey or Ramsey-like, such that $o_{\mathcal{P}}(\mathcal{U}) \geq \alpha$. Then:*

- (1) *For every $U \in \mathcal{U}$, the ultrapower N_U of M_U by U satisfies $o_{\mathcal{P}}(\kappa) \geq \alpha$.*
- (2) *There is a good collection \mathcal{W} with $o_{\mathcal{P}}(\mathcal{W}) = \alpha$ such that $N_W \models o_{\mathcal{P}}(\kappa) = \alpha$ for all $W \in \mathcal{W}$.*

We should not expect an analogue of Theorem 1.2 (1) with equality because we are now dealing with collections of measures instead of a single measure and so Theorem 1.2 (2) is the best possible result.

Theorem 1.3. *Any strongly Ramsey cardinal κ has the maximum Ramsey M-rank $o_{\text{Ram}}(\kappa) = \kappa^+$, any super Ramsey cardinal κ has the maximum strongly Ramsey M-rank $o_{\text{stRam}}(\kappa) = \kappa^+$, and any measurable cardinal κ has the maximum super Ramsey M-rank $o_{\text{supRam}}(\kappa) = \kappa^+$.*

We will show that the new Mitchell order behaves robustly with respect to forcing constructions. We show that extensions with cover and approximation properties cannot create new Ramsey or Ramsey-like cardinals or increase their M-rank. Hamkins showed, in [Ham03], that most large cardinals cannot be created in extensions with cover and approximation properties and we provide several modifications of his techniques to the embeddings characterizing Ramsey and Ramsey-like cardinals. This result is of independent interest since it was not previously known whether Ramsey cardinals can be created in extensions with cover and approximation properties.

Theorem 1.4. *If $V \subseteq V'$ has the δ -cover and δ -approximation properties for some regular cardinal $\delta < \kappa$ of V' , then $o_{\mathcal{P}}^V(\kappa) \geq o_{\mathcal{P}}^{V'}(\kappa)$, where \mathcal{P} is strongly or super Ramsey, and if we additionally assume that $V^\omega \subseteq V$ in V' , then $o_{\text{Ram}}^V(\kappa) \geq o_{\text{Ram}}^{V'}(\kappa)$.*

Using the results about extensions with approximation and cover properties together with new techniques recently developed in Carmody's dissertation [Car15] about softly killing degrees of large cardinals with forcing, we show how to softly kill the M-rank of a Ramsey or Ramsey-like cardinal by forcing.

Theorem 1.5. *If κ has $o_{\mathcal{P}}(\kappa) = \alpha$, where \mathcal{P} is Ramsey or Ramsey-like and $\beta < \alpha$, then there is a cofinality-preserving forcing extension in which $o_{\mathcal{P}}(\kappa) = \beta$.*

Although the general framework of the M-order we have sketched here applies to many smallish large cardinals, we focus in this paper on its application to Ramsey, strongly Ramsey and super Ramsey cardinals. Other instances of it definitely warrant further research.

2. PRELIMINARIES

2.1. Properties of Ramsey and Ramsey-like cardinals. As discussed in the introduction, many large cardinals κ below a measurable cardinal have the prototypical characterization, where for every $A \subseteq \kappa$, there is a weak κ -model M , with

some additional properties, containing A for which there is an M -ultrafilter on κ , with some additional properties, where the additional properties are what distinguishes the different large cardinal properties. Formally, if M is a transitive model of ZFC^- , then an M -ultrafilter is a collection $U \subseteq \mathcal{P}(\kappa)^M$ such that the structure $\langle M, \in, U \rangle$, with a predicate for U , satisfies that U is a normal ultrafilter on κ .³ We can form the ultrapower of a model M by an M -ultrafilter using functions on κ that are elements of M , but the ultrapower may not be well-founded. Let us call U , an M -ultrafilter, δ -*intersecting* for a cardinal δ if every collection of fewer than δ many sets from it has non-empty intersection.⁴ Standard arguments show that the ultrapower of a model M by an ω_1 -intersecting M -ultrafilter is well-founded, but, in contrast with the case of actual ultrafilters, this condition is not necessary.

Many set theoretic constructions use iterated ultrapowers by a measure on a measurable cardinal. If U is an ultrafilter on some set, then the ultrapower (of V) construction with it can be iterated along the ordinals by taking the ultrapower by the image of the previous stage's ultrafilter at successor stages and direct limits at limit stages. Gaifman showed that if an ultrafilter is countably complete, which is equivalent to having a well-founded ultrapower, then all its iterated ultrapowers are well-founded (cf. [Gai74]). For an M -ultrafilter, to be able to define the successor stage ultrafilters in the iterated ultrapower construction, it must be at least partially internal to M , a property that is captured by the notion of weak amenability. An M -ultrafilter is *weakly amenable* if for every $X \in M$ of size at most κ in M , the intersection $X \cap U$ is in M .⁵ Although weak amenability allows us to define all the iterated ultrapowers, it does not have any bearing on their well-foundedness. Kunen showed that being ω_1 -intersecting is sufficient for well-foundedness (cf. [Kun70]), but it is not necessary. Unlike measures on κ , where either all the iterated ultrapowers are well-founded or none are, we will see below that it is consistent to have M -ultrafilters with exactly α -many well-founded iterated ultrapowers for any countable ordinal α .⁶

Recall that κ is weakly compact if and only if $\kappa^{<\kappa} = \kappa$ and every $A \subseteq \kappa$ is contained in a weak κ -model for which there is an M -ultrafilter on κ with a well-founded ultrapower. This characterization can be strengthened in a number of significant ways. For instance, we can assume that M is a κ -model that is elementary in H_{κ^+} and hence reflects V to a certain extent. In fact, we can assume that *every* weak κ -model M has an M -ultrafilter with a well-founded ultrapower. Surprisingly, we cannot strengthen the characterization of Ramsey cardinals in the same fashion. Recall now that κ is Ramsey if and only if every $A \subseteq \kappa$ is contained in a weak κ -model M for which there is a weakly amenable ω_1 -intersecting M -ultrafilter on κ . If we strengthen the characterization to say that M is a κ -model, then we get

³We will always assume that an ultrafilter on a cardinal κ contains the tail sets and therefore is non-principal. It also follows from this assumption that an M -ultrafilter is κ -complete for sequences in M .

⁴In the literature such M -ultrafilters are often called δ -complete which we find confusing because δ -complete ultrafilters are supposed to have the property that the intersection of fewer than δ -many sets in the ultrafilter is itself in the ultrafilter. But in the situation of M -ultrafilters, the intersection may not even be an element of M .

⁵The property is a weakening of the usual definition of amenability because we restrict to X of size at most κ in M .

⁶It follows from Gaifman's arguments in [Gai74] for ultrapowers by a measure that an M -ultrafilter with ω_1 -many well-founded iterated ultrapowers, already has all well-founded iterated ultrapowers.

what Gitman calls a *strongly Ramsey* cardinal, which she showed is a stationary limit of Ramsey cardinals.⁷ Indeed, even if we just assume that M is closed under countable sequences, then we already get a large cardinal, call it *ω -closed Ramsey*, that is a stationary limit of Ramsey cardinals and, which, as we will see in Section 3, has the maximum Ramsey M -rank. Strongly Ramsey cardinals can also be viewed as quite strong because they are limits of the completely Ramsey cardinals defined by Feng in [Fen90]. If we further strengthen the characterization to say that M is elementary in H_{κ^+} , we get a *super Ramsey* cardinal that is in turn a stationary limit of strongly Ramsey cardinals (but weaker than a measurable cardinal). Assuming that *every* weak κ -model M has a weakly amenable ω_1 -intersecting M -ultrafilter turns out to be inconsistent. For details, see [Git11].

The requirement that the M -ultrafilters are weakly amenable already takes us well beyond weak compactness. If every $A \subseteq \kappa$ is contained in a weak κ -model M for which there is a weakly amenable M -ultrafilter with a well-founded ultrapower, then κ is a stationary limit of completely ineffable cardinals, which sit atop a hierarchy of ineffability. The following is a very useful characterization of weak amenability. If U is an M -ultrafilter on κ and $j : M \rightarrow N$ is the ultrapower by U , then U is weakly amenable if and only if $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^N$. Moreover, if $j : M \rightarrow N$ is any embedding with critical point κ and $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^N$, then the M -ultrafilter U obtained from j in the usual way, is weakly amenable. We can stratify weakly amenable M -ultrafilters by degrees of iterability. Let us say that an M -ultrafilter is *α -iterable* if it has α -many well-founded iterated ultrapowers and that it is *iterable* if it is ω_1 -iterable. Gitman defined that a cardinal κ is α -iterable (for $1 \leq \alpha \leq \omega_1$) if every $A \subseteq \kappa$ is contained in a weak κ -model M for which there is a weakly amenable α -iterable M -ultrafilter. Gitman and Welch showed that the α -iterable cardinals form a hierarchy of strength (cf. [GW11]) and Sharpe and Welch showed that an ω_1 -Erdős cardinal is a limit of ω_1 -iterable cardinals (cf. [SW11]).

Suppose that M is a weak κ -model and U is an M -ultrafilter. Consider the submodel $\overline{M} = H_{\kappa^+}^M$ consisting of all sets that have hereditary size at most κ in M . Clearly \overline{M} is itself a weak κ -model and if M was a κ -model then \overline{M} is as well. Also, U is an \overline{M} -ultrafilter and it retains all other relevant properties with respect to \overline{M} that it had with respect to M , such as being weakly amenable, α -iterable, or ω_1 -intersecting. The model \overline{M} is the unique minimal model M_U for U we discussed in the introduction. Note that the sets in M_U are precisely the Mostowski collapses of well-founded binary relations on κ coded by sets in U together with their complements, so that M_U can be recovered from U in any model of a sufficient fragment of set theory. In future arguments, we will only consider M_U -ultrafilters U , where $V_\kappa \in M_U$ so that the ultrapower N_U thinks that $V_{j(\kappa)}$ exists. For such a weakly amenable U , M_U is precisely $H_{\kappa^+}^{N_U}$ and so must be an element of N_U .

It turns out that if U is an iterable M_U -ultrafilter, then U also codes a weak κ -model M_U^* of full ZFC so that it is also an iterable M_U^* -ultrafilter. Specifically, we can take $M_U^* = V_{j(\kappa)}^{N_U}$, where $j : M_U \rightarrow N_U$ is the ultrapower map of M_U by U . The ultrapower map of M_U^* by U is the restriction $j^* : M_U^* \rightarrow N_U^*$ of the ultrapower of N_U by U . If M_U was a κ -model, then so is M_U^* . The embedding j^* has several useful properties, such as $M_U = H_{\kappa^+}^{M_U^*}$, that $M_U^* = V_{j^*(\kappa)}^{N_U^*}$ is in N_U^* , and $M_U^* \prec N_U^*$. The same construction cannot be carried out with a partially

⁷Note that a weakly amenable M -ultrafilter for a κ -model M is automatically ω_1 -intersecting.

iterable M_U -ultrafilter because the iterability of U decreases when you pass to the model M_U^* . Indeed, assuming that there are α -iterable ultrafilters for models of ZFC produces a stronger notion than an α -iterable cardinal.

For forcing constructions with Ramsey cardinals, which we discuss below, we will need to make some additional assumptions on the weak κ -model M . We define that a weak κ -model M is ω -special if it is the union of an elementary chain of (not necessarily transitive) substructures

$$\kappa \in M_0 \prec M_1 \prec \dots \prec M_n \prec \dots$$

for $n < \omega$ such that each $M_n \in M$ and $|M_n|^M = \kappa$.⁸ The ultrapower N of an ω -special weak κ -model M by a weakly amenable M -ultrafilter on κ is ω -special as witnessed by the sequence $\langle X_n \mid n < \omega \rangle$, where

$$X_n = \{j(f)(\kappa) \mid f : \kappa \rightarrow M_n, f \in M_n\},$$

and if $M = M_U$ then M_U^* is ω -special as well. (See Lemmas 2.7 and 2.9 of [GC15].)

Lemma 2.1. *Suppose κ is Ramsey. Then every $A \subseteq \kappa$ is contained in an ω -special weak κ -model M for which there is a weakly amenable ω_1 -intersecting M -ultrafilter on κ .*

Proof. Fix $A \subseteq \kappa$ and choose some weak κ -model \overline{M} containing A for which there is a weakly amenable ω_1 -intersecting \overline{M} -ultrafilter \overline{U} on κ . Let \overline{N} be the ultrapower of \overline{M} by \overline{U} . We can assume that κ is the largest cardinal of \overline{M} and therefore $\overline{M} = H_{\kappa^+}^{\overline{N}}$ is an element of \overline{N} . Working in \overline{N} , let M_0 be any transitive elementary submodel of H_{κ^+} of size κ with $A \in M_0$. Since \overline{N} and \overline{M} have the same subsets of κ , M_0 and hence $U_0 = M_0 \cap \overline{U}$ are in \overline{M} . So working in \overline{N} , we can choose a transitive $M_1 \prec H_{\kappa^+}$ of size κ with $M_0, U_0 \in M_1$. Continuing in this fashion, we obtain a sequence $\langle (M_n, U_n) \mid n < \omega \rangle$. Let $M = \bigcup_{n < \omega} M_n$ and $U = \bigcup_{n < \omega} U_n$. The model M is ω -special as witnessed by the sequence $\langle M_n \mid n < \omega \rangle$ (the M_n are even transitive) and $U = M \cap \overline{U}$ is ω_1 -intersecting and weakly amenable by construction. \square

Since M_U^* is ω -special whenever M_U is, it follows that if κ is Ramsey, then every $A \subseteq \kappa$ is contained in an ω -special weak κ -model $M \models \text{ZFC}$ for which there is a weakly amenable ω_1 -intersecting M -ultrafilter on κ .

2.2. Forcing constructions. Suppose $\mathbb{P} \subseteq V_\kappa$ is a poset and we would like to verify that κ is Ramsey in a forcing extension $V[G]$ by \mathbb{P} . Since every $A \subseteq \kappa$ in $V[G]$ has a \mathbb{P} -name \dot{A} of hereditary size κ in V , \dot{A} together with \mathbb{P} can be put into a weak κ -model M , which ensures that A is in the weak κ -model $M[G]$. Thus, it suffices to show that every ultrapower $j : M \rightarrow N$ of a weak κ -model M by a weakly amenable ω_1 -intersecting M -ultrafilter can be lifted to $j : M[G] \rightarrow N[j(G)]$ so that the lift is the ultrapower by a weakly amenable ω_1 -intersecting $M[G]$ -ultrafilter in $V[G]$. The *lifting criterion* states that j lifts to $j : M[G] \rightarrow N[H]$ with $H = j(G)$ if and only if $j \restriction G \subseteq H$. In this setting, when constructing a generic filter, we usually work with a κ -model M and a poset \mathbb{P} that is $<\kappa$ -closed in M . This suffices for the existence of an M -generic filter for \mathbb{P} . Instead of this approach, which does not apply to weak κ -models, we will use the following *diagonalization criterion*, introduced in [GJ].

⁸In fact we will not require the elementary chain requirement in the definition.

Lemma 2.2 (Diagonalization criterion, [GJ]). *If M is an ω -special weak κ -model and \mathbb{P} is a $\leq\kappa$ -distributive poset in M , then there is an M -generic filter for \mathbb{P} .*

The lift of an ultrapower embedding is always an ultrapower embedding by the ultrafilter W obtained from the lift and we will usually use a direct argument to verify that $M[G]$ and $N[j(G)]$ have the same subsets of κ (which demonstrates weak amenability). To show that W is ω_1 -intersecting, we have the following lemma.

Lemma 2.3 (Gitman, Johnstone [GJ]). *Suppose that M is a weak κ -model and $j : M \rightarrow N$ is the ultrapower by an ω_1 -intersecting M -ultrafilter on κ . Suppose further that $\mathbb{P} \in M$ is a countably closed forcing notion and $G \subseteq \mathbb{P}$ is M -generic. If the ultrapower map j lifts to $j : M[G] \rightarrow N[j(G)]$, then the lift j is the ultrapower map by an ω_1 -intersecting $M[G]$ -ultrafilter in $V[G]$.*

3. THE M-ORDER FOR RAMSEY AND RAMSEY-LIKE CARDINALS

Let's start by recalling and making more precise the definitions we made in the introduction. Let us say that $U \subseteq \mathcal{P}(\kappa)$ of size κ is a *small universe measure* if the collection of Mostowski collapses of well-founded binary relations on κ coded by sets in U and their complements is a weak κ -model M_U such that $V_\kappa \in M_U$ and U is an M_U -ultrafilter. We will write N_U for the Mostowski collapse of the ultrapower of M_U by U , provided that the ultrapower is well-founded. We say that a small universe measure U is a *Ramsey measure* if it is weakly amenable (to M_U) and ω_1 -intersecting, we say that U is a *strong Ramsey measure* if it is weakly amenable and M_U is a κ -model, and we say that U is a *super Ramsey measure* if it is a strong Ramsey measure and $M_U \prec H_{\kappa^+}$. We will carry out all the arguments below for Ramsey cardinals since they are the most complicated, pointing out at the end that analogous or simpler arguments work for strongly Ramsey and super Ramsey cardinals. The interested reader can note along the way where the arguments adapt to other smallish large cardinals, which we do not discuss here.

A first approach to defining a Mitchell-like order for Ramsey cardinals might be to consider ordering Ramsey measures on a fixed Ramsey κ analogously to the Mitchell order on normal measures on a measurable cardinal.

Definition 3.1. Given two Ramsey measures U and W on a cardinal κ , define that $U \triangleleft W$ if $U \in N_W$.

Lemma 3.2. *The relation \triangleleft on Ramsey measures on a cardinal κ is transitive and well-founded.*

Proof. Transitivity is straightforward. To see that the relation is also well-founded, notice that $U \triangleleft W$ implies $j_U(\kappa) < j_W(\kappa)$, where j_U and j_W are the ultrapower maps with respect to U and W . This is so because if $U \in N_W$ then also $j_U, N_U \in N_W$ and $N_W \models |U| = |N_U|$. Now, $|U|^{N_W} \leq (2^\kappa)^{N_W} < j_W(\kappa)$ since $j_W(\kappa)$ is inaccessible in N_W . Since $j_U(\kappa)$ is an element of N_U , a transitive substructure of N_W , we must have $j_U(\kappa) < j_W(\kappa)$. The well-foundedness of \triangleleft now follows since an infinite decreasing chain of Ramsey measures would yield an infinite decreasing chain of ordinals. \square

We should also notice that no Ramsey measure U can have more than κ predecessors in the order \triangleleft , since the ultrapower N_U has cardinality κ .

The order \triangleleft on Ramsey measures is an interesting object in its own right, but it is not useful for defining degrees of Ramsey cardinals with the intention to capture

the extent to which Ramseyness is reflected below κ because if κ is Ramsey then there are already Ramsey measures of all possible ranks $\alpha < \kappa^+$.

Lemma 3.3. *If κ is Ramsey and $\alpha < \kappa^+$, then there is a Ramsey measure on κ of rank at least α in the \triangleleft -order.*

Proof. Suppose inductively that for all $\beta < \alpha$, there is a Ramsey measure U_β on κ whose rank in the \triangleleft -order is at least β . Let U be some Ramsey measure on κ such that $\{U_\beta \mid \beta < \alpha\} \subseteq M_U$, which is possible since this set has hereditary size κ . Clearly the rank of U in \triangleleft is at least α . \square

The same analysis holds for strongly Ramsey and super Ramsey measures. It is to be expected that ordering the small universe measures is not the right analogue of the Mitchell order for smaller large cardinals because such a cardinal is characterized by the existence of many and not just one such measure. This brings us back to the definition of the M-order on collections of small universe measures, which we restate here again in full generality before going to back concrete arguments for Ramsey cardinals. Recall that a \mathcal{P} -measure (for some large cardinal property \mathcal{P} such as Ramsey, strongly Ramsey, etc.) is *A-good* for $A \subseteq \kappa$ if $A \in M_U$. We call a collection of \mathcal{P} -measures a *witness collection* if it contains at least one *A-good* \mathcal{P} -measure for every $A \subseteq \kappa$. Then κ has property \mathcal{P} if and only if there is a witness collection of \mathcal{P} -measures.

Definition 3.4. (M-order) Suppose that κ is a large cardinal with property \mathcal{P} having a suitable embedding characterization. Given two witness collections \mathcal{U} and \mathcal{W} of \mathcal{P} -measures, we define that $\mathcal{U} \triangleleft \mathcal{W}$ whenever

- (1) For every $W \in \mathcal{W}$ and $A \subseteq \kappa$ in the ultrapower N_W of M_W by W , there is an *A-good* $U \in \mathcal{U} \cap N_W$ such that N_W agrees that U is an *A-good* \mathcal{P} -measure on κ .
- (2) $\mathcal{U} \subseteq \bigcup_{W \in \mathcal{W}} N_W$.

Lemma 3.5. *The \triangleleft -order on witness collections of Ramsey measures on a cardinal κ is transitive and well-founded.*

Proof. First, we show transitivity. Suppose that $\mathcal{U}, \mathcal{W}, \mathcal{Z}$ are witness collections of Ramsey measures on κ such that $\mathcal{U} \triangleleft \mathcal{W}$ and $\mathcal{W} \triangleleft \mathcal{Z}$. We have $\mathcal{U} \subseteq \bigcup_{W \in \mathcal{W}} N_W \subseteq \bigcup_{Z \in \mathcal{Z}} N_Z$, where the first inclusion follows by definition and second inclusion follows because $\mathcal{W} \subseteq \bigcup_{Z \in \mathcal{Z}} N_Z$ and if $W \in \mathcal{W}$ is in some N_Z , then N_Z has N_W as well. This verifies requirement (2) in showing that $\mathcal{U} \triangleleft \mathcal{Z}$, and now we verify (1). If $Z \in \mathcal{Z}$ and $A \in N_Z$, then since $\mathcal{W} \triangleleft \mathcal{Z}$, there is $W \in N_Z \cap \mathcal{W}$ with $A \in N_W$, but then, since $\mathcal{U} \triangleleft \mathcal{W}$, there is $U \in N_W \cap \mathcal{U} \subseteq N_Z \cap \mathcal{U}$ with $A \in N_U$. Thus, $\mathcal{U} \triangleleft \mathcal{Z}$.

Next, suppose towards a contradiction that \triangleleft is ill-founded for witness collections of Ramsey measures on κ and fix a \triangleleft -descending sequence

$$\mathcal{U}_0 \triangleright \mathcal{U}_1 \triangleright \cdots \triangleright \mathcal{U}_n \triangleright \cdots$$

of witness collections. Let U_0 be any element of \mathcal{U}_0 . Since $\mathcal{U}_1 \triangleleft \mathcal{U}_0$, then N_{U_0} has some element U_1 of \mathcal{U}_1 , and so $U_1 \triangleleft U_0$ in the ordering on Ramsey measures. Continuing in the same manner, we obtain a descending sequence

$$U_0 \triangleright U_1 \triangleright \cdots \triangleright U_n \triangleright \cdots$$

in the \triangleleft -order on Ramsey measures, which is impossible by Lemma 3.2. \square

The lemma implies that we can assign to each witness collection \mathcal{U} of Ramsey measures on κ its rank $o_{\text{Ram}}(\mathcal{U})$ in the order \triangleleft . We can then let

$$o_{\text{Ram}}(\kappa) = \{o_{\text{Ram}}(\mathcal{U}) \mid \mathcal{U} \text{ is a witness collection of Ramsey measures on } \kappa\}.$$

We define ranks for strongly and super Ramsey cardinals in similar fashion.

The defining property of the Mitchell order is that a normal measure U on κ has rank α if and only if $\text{Ult}(V, U)$ satisfies $o(\kappa) = \alpha$. The analogous result for the M-order on witness collections of Ramsey measures on κ will be that $o_{\text{Ram}}(\mathcal{U}) \geq \alpha$ if and only if for every $U \in \mathcal{U}$, $N_U \models o_{\text{Ram}}(\kappa) \geq \alpha$. It is not feasible to obtain equality because a witness collection \mathcal{U} of rank α can easily have elements U with $N_U \models o_{\text{Ram}}(\kappa) > \alpha$. Still we will be able to show that “well-behaved” collections always exist: if $\alpha < o_{\text{Ram}}(\kappa)$, then there is some witness collection \mathcal{W} with $o_{\text{Ram}}(\mathcal{W}) = \alpha$ such that $N_W \models o_{\text{Ram}}(\kappa) = \alpha$ for every $W \in \mathcal{W}$.

A subtle issue that arises when trying to prove that the rank of a witness collection corresponds to the rank of κ in the ultrapowers of its measures in the case of Ramsey cardinals (but does not arise for strongly Ramsey or super Ramsey cardinals) is that the ultrapower of a weak κ -model can already be wrong about whether something is a Ramsey measure or not since in most cases it is missing countable sequences. To prove the result we will temporarily use a stronger notion of Ramsey measure, which will be absolute for transitive ZFC^- -models, and show that the two notions give the same M-rank.

Let us say that an M_U -ultrafilter U on κ is a *certified Ramsey measure* if it is weakly amenable and there is some unbounded $I \subseteq \kappa$ such that $X \in U$ if and only if X contains a tail of I , in which case we say that I *certifies* U . Clearly, every certified Ramsey measure is a Ramsey measure because it is even κ -intersecting (every sequence of $< \kappa$ -many sets in U has a non-empty intersection) and certified Ramsey measures have the advantage of being absolute between transitive models of set theory. In fact, a standard proof that Ramsey cardinals have Mitchell’s characterization (see [Dod82] or a more detailed exposition in [Git07]) actually produces for every $A \subseteq \kappa$, an A -good certified Ramsey measure. Briefly, the proof uses the notion of a *good set of indiscernibles* for a structure $L_\kappa[A]$ with $A \subseteq \kappa$ and shows that if κ is Ramsey, then for every $A \subseteq \kappa$, there is a good set I of indiscernibles for $L_\kappa[A]$ of size κ . The indiscernibles in I are then used to construct a weak κ -model M (with the largest cardinal κ) and a weakly amenable ω_1 -intersecting M -ultrafilter that is certified by I . Given a witness collection \mathcal{U} of certified Ramsey measures, let $o_{\text{Ram}}^*(\mathcal{U})$ denote the rank of \mathcal{U} in the \triangleleft -order restricted to witness collections of certified Ramsey measures and let $o_{\text{Ram}}^*(\kappa)$ be the supremum of the ranks of all such \mathcal{U} .

For inductive arguments about the M-rank of Ramsey or Ramsey-like cardinals, we will often need to know that the results hold not just in V , but more generally in transitive set models of ZFC^- that know enough about the cardinal. If \mathcal{M} is a transitive model of ZFC^- and κ is a cardinal in \mathcal{M} , we will say that \mathcal{M} is *practical* for κ if $V_{\kappa+3}^{\mathcal{M}}$ exists (this ensures that the model can put together all witness collections in order to rank them). We formulate the following several results about Ramsey and Ramsey-like cardinals for practical models and note that the results also hold for V , since we may always work with a sufficiently large H_λ for which all of the notions are absolute.

Lemma 3.6. *If α is an ordinal and \mathcal{M} is practical for κ , then, in \mathcal{M} , a witness collection \mathcal{U} of certified Ramsey measures on κ has $o_{\text{Ram}}^*(\mathcal{U}) \geq \alpha$ if and only if $N_U \models o_{\text{Ram}}^*(\kappa) \geq \alpha$ for all $U \in \mathcal{U}$.*

Proof. We will argue by induction on α . The case $\alpha = 0$ is trivial. So suppose that the statement is true for all $0 \leq \beta < \alpha$. Fix an \mathcal{M} practical for κ and work in \mathcal{M} .

In one direction, fix a witness collection \mathcal{U} of certified measures on κ with $o_{\text{Ram}}^*(\mathcal{U}) \geq \alpha$. Let $U \in \mathcal{U}$. We must show that N_U has witness collections of certified Ramsey measures of all ranks $\beta < \alpha$. Fix $\beta < \alpha$. Since $o_{\text{Ram}}^*(\mathcal{U}) \geq \alpha$, there must be some witness collection $\mathcal{W} \triangleleft \mathcal{U}$ of certified Ramsey measures with $o_{\text{Ram}}^*(\mathcal{W}) = \beta$. By the inductive hypothesis applied to \mathcal{M} , for all $W \in \mathcal{W}$, we have $N_W \models o_{\text{Ram}}^*(\kappa) \geq \beta$. Therefore, since $\mathcal{W} \triangleleft \mathcal{U}$, there is for every $A \subseteq \kappa$ in N_U some A -good certified Ramsey measure $W \in N_U$ with $N_W \models o_{\text{Ram}}^*(\kappa) \geq \beta$ and so N_U , by collecting these together, has a witness collection $\overline{\mathcal{W}}$ of certified Ramsey measures such that $N_W \models o_{\text{Ram}}^*(\kappa) \geq \beta$ holds for all $W \in \overline{\mathcal{W}}$. But then, by applying our inductive hypothesis to N_U , we have that $o_{\text{Ram}}^*(\overline{\mathcal{W}}) \geq \beta$ in N_U . This completes the proof in one direction.

In the other direction, suppose that \mathcal{U} is a witness collection of certified Ramsey measures such that $N_U \models o_{\text{Ram}}^*(\kappa) \geq \alpha$ for all $U \in \mathcal{U}$. We must show that for all $\beta < \alpha$, there is a witness collection \mathcal{W} of certified Ramsey measures with $\mathcal{W} \triangleleft \mathcal{U}$ and $o_{\text{Ram}}^*(\mathcal{W}) \geq \beta$. Fix $\beta < \alpha$. For each $U \in \mathcal{U}$, we can fix some \mathcal{W}_U , which N_U thinks is a witness collection of certified Ramsey measures of rank at least β , and let $\mathcal{W} = \bigcup_{U \in \mathcal{U}} \mathcal{W}_U$. Since all our measures are certified, \mathcal{W} is a witness collection of certified Ramsey measures and we have arranged that $\mathcal{W} \triangleleft \mathcal{U}$. If $W \in \mathcal{W}$, then $W \in \mathcal{W}_U$ for some $U \in \mathcal{U}$ and therefore, since $N_U \models o_{\text{Ram}}^*(\mathcal{W}_U) \geq \beta$, we get $N_W \models o_{\text{Ram}}^*(\kappa) \geq \beta$ by applying the inductive hypothesis to N_U . Thus, by the inductive hypothesis, $o_{\text{Ram}}^*(\mathcal{W}) \geq \beta$, which establishes this direction. \square

Note that the fact that the Ramsey measures we are working with are certified only came into play in the second part of the proof. Essentially, being a Ramsey measure is downward absolute. On the other hand, we would have run into trouble in the second part if we had built the collection \mathcal{W} using only ordinary Ramsey measures, since N_U and \mathcal{M} might disagree on whether a given filter is ω_1 -intersecting. This observation will be important when we revisit this proof in Theorem 3.7.

The desired result, which is the same lemma for witness collections of Ramsey measures, will follow once we establish that $o_{\text{Ram}}^*(\kappa) = o_{\text{Ram}}(\kappa)$. First, we have to review a few basic facts which we will use now and in later sections.

Observe that if U is a Ramsey measure, then the intersection of any countably many sets in U has size κ because if the intersection was bounded by $\alpha < \kappa$, we could add $\kappa \setminus \alpha$ to the sets being intersected (recall that all tails of κ are in U by assumption) and violate the ω_1 -intersecting property. Another useful fact is that for every ordinal $\kappa \leq \alpha < \kappa^+$, if E is a well-ordering of κ in order-type α , then there is a single function $g^E : \kappa \rightarrow \kappa$ such that whenever U is an M -ultrafilter with a well-founded ultrapower and $E \in M$, then $g^E \in M$ and $[g^E]_U = \alpha$ in the ultrapower. We call such g^E a *representing function* for α and can define it by simply letting $g^E(\xi)$ be the order-type of $E \upharpoonright \xi \times \xi$.

Theorem 3.7. *Let α be an ordinal and \mathcal{M} practical for κ . Then the following hold in \mathcal{M} :*

- (1) a witness collection \mathcal{U} of Ramsey measures on κ has $o_{\text{Ram}}(\mathcal{U}) \geq \alpha$ if and only if $N_U \models o_{\text{Ram}}(\kappa) \geq \alpha$ for all $U \in \mathcal{U}$;
- (2) if there is a witness collection \mathcal{U} of Ramsey measures on κ with $o_{\text{Ram}}(\mathcal{U}) \geq \alpha$, then there is also a witness collection \mathcal{U}^* of certified Ramsey measures with $o_{\text{Ram}}(\mathcal{U}^*) \geq \alpha$.

Note that we used the simple M-rank in both cases in part (2) of the lemma and not the M-rank restricted to witness collections of certified measures.

Proof. We will prove both parts of the statement simultaneously by induction on α . For the base case $\alpha = 0$ part (1) is trivial and part (2) follows because, as we already noted, if κ is Ramsey, then for every $A \subseteq \kappa$, there is an A -good certified Ramsey measure. So suppose inductively that the statement holds for all $0 \leq \beta < \alpha$. Fix an \mathcal{M} practical for κ and work in \mathcal{M} .

Let us first show that part (1) holds for α by mirroring the proof of Lemma 3.6. The forward direction goes through exactly as in the proof of Lemma 3.6, since, as we noted after that proof, the fact that the Ramsey measures were certified played no part in this particular argument. For the converse, suppose that \mathcal{U} is a witness collection of Ramsey measures such that $N_U \models o_{\text{Ram}}(\kappa) \geq \alpha$ for all $U \in \mathcal{U}$. Fix a $\beta < \alpha$. For each $U \in \mathcal{U}$ we can fix \mathcal{W}_U which N_U thinks is a collection of Ramsey measures of rank at least β and let $\mathcal{W} = \bigcup_{U \in \mathcal{U}} \mathcal{W}_U$. We would like to say that \mathcal{W} is a witness collection of Ramsey measures, but this need not be the case if we are working with arbitrary (noncertified) Ramsey measures. Instead, we apply part (2) of the induction hypothesis to each N_U to replace each \mathcal{W}_U with a witness collection \mathcal{W}_U^* of certified Ramsey measures satisfying $o_{\text{Ram}}(\mathcal{W}_U^*) \geq \beta$ in N_U . If we now let $\mathcal{W}^* = \bigcup_{U \in \mathcal{U}} \mathcal{W}_U^*$, this actually is a witness collection of Ramsey measures and, again, $\mathcal{W}^* \triangleleft \mathcal{U}$. The rest of the argument proceeds as before: if $W \in \mathcal{W}^*$, then $W \in \mathcal{W}_U^*$ for some $U \in \mathcal{U}$ and therefore, since $N_U \models o_{\text{Ram}}(\mathcal{W}_U^*) \geq \beta$, we get $N_W \models o_{\text{Ram}}(\kappa) \geq \beta$ by applying part (1) of the induction hypothesis to N_U . Thus, by part (1) of the induction hypothesis again, $o_{\text{Ram}}(\mathcal{W}^*) \geq \beta$. Altogether, this shows that $o_{\text{Ram}}(\mathcal{U}) \geq \alpha$.

Now we move on to show that part (2) holds for α . Suppose that there is a witness collection \mathcal{U} of Ramsey measures on κ with $o_{\text{Ram}}(\mathcal{U}) \geq \alpha$. We need to show that there is a witness collection \mathcal{U}^* of certified Ramsey measures on κ with $o_{\text{Ram}}(\mathcal{U}^*) \geq \alpha$. By what we just argued it follows that $N_U \models o_{\text{Ram}}(\kappa) \geq \alpha$ for all $U \in \mathcal{U}$. The next step is to replace each U with some U^* , where U^* is certified and N_{U^*} also satisfies that $o_{\text{Ram}}(\kappa) \geq \alpha$. For this, we need to look more closely at how a good set I of indiscernibles for $L_\kappa[A]$ is constructed.

For every $A \subseteq \kappa$, there is an associated club C_A in κ and a regressive function $f_A : [C_A]^{<\omega} \rightarrow \kappa$ such that any homogeneous set for f_A is a good set of indiscernibles for $L_\kappa[A]$. The club C_A and function f_A are defined simply enough from A that any transitive model of ZFC^- containing A also contains C_A and f_A (for details, see [Git07], chapter 2). Given an A -good Ramsey measure U , we will find a homogeneous set I of size κ for f_A by showing that for each $n < \omega$, the restriction $f_n : [C_A]^n \rightarrow \kappa$ of f_A has a homogeneous set in U and using the ω_1 -intersecting property of U . Since U is weakly amenable, we can define the finite product M_U -ultrafilters U^n for $n < \omega$ (where $U^1 = U$) and since all iterated ultrapowers of U are well-founded and the ultrapower by U^n is isomorphic to the n^{th} -iterated ultrapower of U , it follows that all ultrapower maps $j_{U^n} : M_U \rightarrow N_{U^n}$ are embeddings into transitive models. Standard facts about product ultrafilters also tell us that a

set $B \subseteq \kappa^n$ is in U^n if and only if $\langle \kappa, j_U(\kappa), j_{U^2}(\kappa), \dots, j_{U^{n-1}}(\kappa) \rangle \in j_{U^n}(B)$. Now fix $n < \omega$ and consider f_n . The set $[C_A]^n$ is in U^n , since every club is in U , and so we can let $j_{U^n}(f_n)(\kappa, j_U(\kappa), j_{U^2}(\kappa), \dots, j_{U^{n-1}}(\kappa)) = \xi$, where we must have $\xi < \kappa$ since $j_{U^n}(f_n)$ is regressive by elementarity. It follows that the set

$$X_n = \{ \langle \xi_1, \xi_2, \dots, \xi_n \rangle \in [C]^n \mid f_n(\xi_1, \xi_2, \dots, \xi_n) = \xi \}$$

is in U . By properties of product ultrafilters, there is a set $\bar{X}_n \in U$ such that every sequence $\langle \xi_1, \xi_2, \dots, \xi_n \rangle \in [C]^n$ with $\xi_i \in \bar{X}_n$ is in X_n . Clearly each \bar{X}_n is homogeneous for f_n and so we can intersect all the \bar{X}_n to obtain a homogeneous set I of size κ for f_A . Note that we can further refine I by adding some other sets in U to the intersection.

Now fix some $A \subseteq \kappa$ and find an $\{A, E\}$ -good Ramsey measure U in \mathcal{U} , where E is some well-order of κ of order-type α , so that we have the representing function g^E in M_U (if $\alpha < \kappa$ we can use a constant function instead of g^E and omit E from the following discussion). Since $N_U \models o_{\text{Ram}}(\kappa) \geq \alpha$, the set

$$X = \{ \xi < \kappa \mid o_{\text{Ram}}(\xi) \geq g^E(\xi) \}$$

is in U . This is crucial to the ensuing construction. Let $A^* \subseteq \kappa$ code the triple $\{A, E, V_\kappa\}$. Now we consider the regressive function $f_{A^*} : [C_{A^*}]^{<\omega} \rightarrow \kappa$ and construct a good set I_{A^*} of indiscernibles for $L_\kappa[A^*]$ by intersecting the sets \bar{X}_n , homogeneous for f_n , together with X . This ensures that $I_{A^*} \subseteq X$. Using I_{A^*} , we construct a certified Ramsey measure U^* with $A, E, V_\kappa \in M_{U^*}$, which is certified by I_{A^*} . Note that X is an element of M_{U^*} because it is definable over V_κ from E and so it must be the case that $X \in U^*$. But since $E \in M_{U^*}$, it follows that $[g^E] = \alpha$ in the ultrapower N_{U^*} and so $N_{U^*} \models o_{\text{Ram}}(\kappa) \geq \alpha$. Thus, we have succeeded in finding for every $A \subseteq \kappa$, an A -good certified Ramsey measure U^* such that $N_{U^*} \models o_{\text{Ram}}(\kappa) \geq \alpha$. Let \mathcal{U}^* be the witness collection consisting of these U^* . By part (1) above, $o_{\text{Ram}}(\mathcal{U}^*) \geq \alpha$, which completes the argument. \square

The proofs of analogous results for witness collections of strong Ramsey or super Ramsey measures are even easier, in that we do not even need to introduce certified measures. A κ -model is always correct about a set being a strong Ramsey measure and a κ -model that is elementary in H_{κ^+} is always correct about a set being a super Ramsey measure: if $M \prec H_{\kappa^+}$ is a κ -model and U is a weakly amenable M -ultrafilter, then M is the H_{κ^+} of the ultrapower N and therefore if $\bar{M} \prec H_{\kappa^+}^N$, then $\bar{M} \prec H_{\kappa^+}$.

A more direct approach to defining the rank of a Ramsey (or Ramsey-like) cardinal, without introducing the order on the witness collections, would be as follows. Define that the Ramsey rank of κ is 0 if κ is not Ramsey, that the Ramsey rank of κ is ≥ 1 if κ is Ramsey, and now inductively that the Ramsey rank of κ is $\geq \alpha$ if for every $A \subseteq \kappa$ and $\beta < \alpha$, there is an A -good Ramsey measure U such that the Ramsey rank of κ in N_U is $\geq \beta$. Finally, define that the rank of κ is exactly α if it is $\geq \alpha$, but it is not $\geq \alpha + 1$. As a corollary of Theorem 3.7, we get that the M-rank is precisely the Ramsey rank we just described.

Corollary 3.8. *If α is an ordinal and \mathcal{M} is practical for κ , then, in \mathcal{M} , we have $o_{\text{Ram}}(\kappa) \geq \alpha$ if and only if for every $A \subseteq \kappa$ and every $\beta < \alpha$, there is an A -good Ramsey measure W on κ with $N_W \models o_{\text{Ram}}(\kappa) \geq \beta$. The same result holds for strongly Ramsey and super Ramsey measures.*

Corollary 3.8 allows us to calculate $o_{\text{Ram}}(\kappa)$ inside H_{κ^+} and confirms the intuition that objects in H_{κ^+} should suffice to compute the rank of a Ramsey or Ramsey-like cardinal. An important advantage of this alternative description of the M -rank is that it is meaningful even in models of set theory which are not practical for κ , e.g. in κ -models where $\mathcal{P}(\kappa)$ does not exist. Such models might contain many Ramsey measures on κ but cannot collect them into a witnessing collection. Consequently, the M -rank, as originally defined, of κ in such a model would be 0, but computing it in this alternative way might give nontrivial values.

Next, as promised, we show that there are always “well-behaved” witness collections of Ramsey (strongly Ramsey, super Ramsey) measures.

Theorem 3.9. *If α is an ordinal and \mathcal{M} is practical for κ , then, in \mathcal{M} , whenever $o_{\text{Ram}}(\kappa) > \alpha$, there is a witness collection \mathcal{U} of Ramsey measures on κ with $o_{\text{Ram}}(\mathcal{U}) = \alpha$ such that $N_U \models o_{\text{Ram}}(\kappa) = \alpha$ for all $U \in \mathcal{U}$. The same result holds for strongly Ramsey and super Ramsey measures.*

Proof. As usual, we prove the result for Ramsey measures. Fix an \mathcal{M} practical for κ and work in \mathcal{M} . Suppose that \mathcal{W} is a witness collection of Ramsey measures on κ with $o_{\text{Ram}}(\mathcal{W}) = \alpha$. If for every $A \subseteq \kappa$, there is some A -good Ramsey measure U such that $N_U \models o_{\text{Ram}}(\kappa) = \alpha$, then we can let \mathcal{U} be the witness collection of such Ramsey measures, one for every A , and by Theorem 3.7, we would have $o_{\text{Ram}}(\mathcal{U}) = \alpha$. Thus, we can suppose towards a contradiction that there is some $A \subseteq \kappa$ such that for every A -good Ramsey measure U , if $N_U \models o_{\text{Ram}}(\kappa) \geq \alpha$, then $N_U \models o_{\text{Ram}}(\kappa) > \alpha$. It follows from this assumption and Theorem 3.7 that there is for every $B \subseteq \kappa$ a $\{A, B\}$ -good Ramsey measure $W \in \mathcal{W}$ with $N_W \models o_{\text{Ram}}(\kappa) > \alpha$. Let \mathcal{W}_0 be a witness collection consisting of one such Ramsey measure for every B . Thus, if $W \in \mathcal{W}_0$, then N_W has what it thinks is a witness collection of certified Ramsey measures of rank greater than α by Theorem 3.7 and so, in particular, N_W has, for any $B \subseteq \kappa$ in N_W , an $\{A, B\}$ -good certified measure \bar{W} with the property that $N_{\bar{W}} \models o_{\text{Ram}}(\kappa) > \alpha$. Let \mathcal{W}_1 be the witness collection formed by putting together all such certified measures from all N_W for $W \in \mathcal{W}_0$. By construction $\mathcal{W}_1 \triangleleft \mathcal{W}_0$. But \mathcal{W}_1 has the same property as \mathcal{W}_0 , namely that for every $W \in \mathcal{W}_1$, $N_W \models o_{\text{Ram}}(\kappa) > \alpha$. Thus, we can repeat the process to construct $\mathcal{W}_2 \triangleleft \mathcal{W}_1$ with the same property and in this way obtain a descending infinite sequence in \triangleleft , which is impossible. \square

The theorem allows us to obtain the following sharpened version of Corollary 3.8.

Corollary 3.10. *If α is an ordinal and \mathcal{M} is practical for κ , then, in \mathcal{M} , we have $o_{\text{Ram}}(\kappa) \geq \alpha$ if and only if every $A \subseteq \kappa$ and every $\beta < \alpha$, there is an A -good Ramsey measure W on κ with $N_W \models o_{\text{Ram}}(\kappa) = \beta$. The same result holds for strongly Ramsey and super Ramsey measures.*

We end the discussion of basic properties of the M -order on witness collections by showing that strongly Ramsey cardinals have the maximum Ramsey rank, super Ramsey cardinals have the maximum strongly Ramsey rank and measurable cardinals have the maximum super Ramsey rank.

Theorem 3.11.

- (1) *If κ is strongly Ramsey, then $o_{\text{Ram}}(\kappa) = \kappa^+$.*
- (2) *If κ is super Ramsey, then $o_{\text{stRam}}(\kappa) = \kappa^+$.*
- (3) *If κ is measurable, then $o_{\text{supRam}}(\kappa) = \kappa^+$.*

Proof. Let us introduce an intermediate large cardinal property between Ramsey and super Ramsey cardinals by removing the κ -model assumption from the definition of super Ramsey cardinals. Call a cardinal *weakly super Ramsey* if for every $A \subseteq \kappa$, there is a weak κ -model $M \prec H_{\kappa^+}$ containing A for which there is a weakly amenable ω_1 -intersecting M -ultrafilter on κ . We will argue that a weakly super Ramsey κ must have maximum Ramsey rank. Suppose not, meaning that $o_{\text{Ram}}(\kappa) = \alpha < \kappa^+$. By Corollary 3.10 there is in H_{κ^+} , for every $\beta < \alpha$, an A -good Ramsey measure W with $N_W \models o_{\text{Ram}}(\kappa) = \beta$. Fix $A \subseteq \kappa$ and let $M \prec H_{\kappa^+}$ be a weak κ -model containing A and α for which there is a weakly amenable ω_1 -intersecting M -ultrafilter on κ . Let N be the ultrapower of M by U . If $\beta < \alpha$, then, by elementarity, M satisfies that for every $B \subseteq \kappa$, there is a Ramsey measure W on κ with $N_W \models o_{\text{Ram}}(\kappa) = \beta$, and so N must satisfy this as well. But then Corollary 3.10 implies $N \models o_{\text{Ram}}(\kappa) \geq \alpha$. Thus, we have shown that for every $A \subseteq \kappa$, there is an A -good Ramsey measure U on κ with $N_U \models o_{\text{Ram}}(\kappa) = \alpha$, which means that $o_{\text{Ram}}(\kappa) > \alpha$, contradicting our assumption.

Let κ be strongly Ramsey. We will show that for every $\alpha < \kappa^+$, there is a witness collection of Ramsey measures on κ of rank α . Fix $A \subseteq \kappa$ and let U be an $\{A, \alpha\}$ -good strong Ramsey measure on κ . We will now argue that κ is weakly super Ramsey in N_U . Fix $B \subseteq \kappa$ in N_U . Using the construction from the proof of Lemma 2.1, we obtain a sequence $\langle (M_n, W_n) \mid n < \omega \rangle$ such that $B \in M = \bigcup_{n < \omega} M_n$ is elementary in M_U and $W = \bigcup_{n < \omega} W_n$ is a weakly amenable M -ultrafilter on κ . Since N_U is a κ -model both M and W are in N_U . Thus, we have verified that κ is weakly super Ramsey in N_U , and so it follows that $N_U \models o_{\text{Ram}}(\kappa) = \kappa^+$, so in particular $N_U \models o_{\text{Ram}}(\kappa) \geq \alpha$. But this means that for every $A \subseteq \kappa$, there is an A -good Ramsey measure U with $N_U \models o_{\text{Ram}}(\kappa) \geq \alpha$, from which it follows that there is a witness collection of Ramsey measures on κ of rank α , as required.

To show that strong Ramsey cardinals have maximum strong Ramsey rank, we just mimic the argument that weakly super Ramsey cardinals have maximum Ramsey rank. To show that measurable cardinals have maximum super Ramsey rank, we use that measurable cardinals are super Ramsey and repeat the same argument. \square

Note that we didn't need that κ is strongly Ramsey in the argument that $o_{\text{Ram}}(\kappa) = \kappa^+$, but merely that κ is ω -closed Ramsey ($M^\omega \subseteq M$), which gives a lower bound on the strength of having maximum Ramsey rank. In fact, the proof shows that ω -closed Ramsey cardinals are stationary limits of Ramsey cardinals of maximal Ramsey rank.

4. EXTENSIONS WITH COVER AND APPROXIMATION PROPERTIES CANNOT INCREASE RAMSEY OR RAMSEY-LIKE RANK

In [Ham03], Hamkins developed general techniques to show that if $V \subseteq V'$ has the δ -cover and δ -approximation properties for some regular cardinal δ of V' , then for most large cardinal properties V' cannot have new large cardinals of that type above δ . The techniques cannot be applied directly to Ramsey or Ramsey-like cardinals because, for the smaller large cardinals, they require embeddings to exist for all transitive models of size κ (as in the case of weakly compact cardinals), and in particular for all κ -models, which we know is not the case for Ramsey or Ramsey-like cardinals. Nevertheless, we will be able to adapt the machinery used in the proofs of theorems in [Ham03] to the situation of our cardinals. We will show that

if $V \subseteq V'$ has the δ -cover and δ -approximation properties (for some regular δ of V') and $\kappa > \delta$ has Ramsey (or Ramsey-like) rank α in V' , then it had at least rank α in V . The significance of the result lies in applying it to forcing extensions to show that no new Ramsey or Ramsey-like cardinals of any rank were created. Although it is easy to show that Ramsey cardinals cannot be created by small forcing, it was not previously known whether the result generalized to all extensions with cover and approximation properties. We begin by recalling the definition of cover and approximation properties and their connection to forcing extensions.

Definition 4.1 (Hamkins [Ham03]). Suppose $V \subseteq V'$ are transitive (set or class) models of (some fragment of) ZFC and δ is a cardinal in V' .

- (1) The pair $V \subseteq V'$ satisfies the δ -cover property if for every $X \in V'$ with $X \subseteq V$ and $|X|^{V'} < \delta$, there is $Y \in V$ with $X \subseteq Y$ and $|Y|^V < \delta$.
- (2) The pair $V \subseteq V'$ satisfies the δ -approximation property if whenever $X \in V'$ with $X \subseteq V$ and $X \cap x \in V$ for every x of size less than δ in V , then $X \in V$.

If \mathbb{P} is a forcing notion of size at most δ , then the pair $V \subseteq V[G]$, where G is V -generic for \mathbb{P} , has the δ^+ -cover and δ^+ -approximation properties. We say that a poset \mathbb{P} has a *closure point* at a cardinal δ if \mathbb{P} factors as $\mathbb{R} * \dot{\mathbb{Q}}$, where \mathbb{R} is nontrivial⁹ of size at most δ and $\Vdash_{\mathbb{R}} \dot{\mathbb{Q}}$ is strategically $\leq \delta$ -closed. We then have:

Theorem 4.2 (Hamkins). *If \mathbb{P} is a forcing notion with a closure point at δ , then the pair $V \subseteq V[G]$ satisfies the δ^+ -cover and δ^+ -approximation properties for any forcing extension $V[G]$ by \mathbb{P} .*

Thus, we will be able to show that a large class of forcing notions, namely those with a closure point less than or equal to the first inaccessible cardinal (or in fact much higher), cannot create new Ramsey or Ramsey-like cardinals of any rank.

4.1. Strongly Ramsey and super Ramsey cardinals. Most of the work in showing that strong Ramsey rank cannot increase in extensions with cover and approximation properties goes into showing that strongly Ramsey cardinals cannot be created in such extensions. Once again, to carry out the inductive arguments, we will need the statements to be formulated in terms of practical models, with the hypothesis that \mathcal{M} and \mathcal{M}' are practical for κ and that the pair $\mathcal{M} \subseteq \mathcal{M}'$ has the δ -cover and δ -approximation properties and the same ordinals.

Lemma 4.3. *Suppose the pair $V \subseteq V'$ satisfies the δ -cover and δ -approximation properties and has the same ordinals. Let $\lambda \geq \delta$ be a regular cardinal in V' . Then the pair $H_\lambda^V \subseteq H_\lambda^{V'}$ also satisfies the δ -cover and δ -approximation properties and has the same ordinals.*

Proof. Suppose that α is an ordinal in $H_\lambda^{V'}$, meaning that $\alpha < \lambda$. It follows that α has size less than λ in V as well (otherwise, λ would not be a cardinal in V'), and hence $\alpha \in H_\lambda^V$, so H_λ^V and $H_\lambda^{V'}$ have the same ordinals. If $X \subseteq H_\lambda^V$ is a set of size less than δ in $H_\lambda^{V'}$, then, by the δ -covering property of $V \subseteq V'$ there is a Y of size less than δ in V with $X \subseteq Y$. Let $\bar{Y} = Y \cap H_\lambda^V$. Then \bar{Y} is a set of size less than δ each of whose elements has transitive size less than λ , and so by regularity of λ we get $\bar{Y} \in H_\lambda^V$, showing that $H_\lambda^V \subseteq H_\lambda^{V'}$ satisfies the δ -cover property. Finally, suppose that $X \subseteq H_\lambda^V$ is in $H_\lambda^{V'}$ and $x \cap X \in H_\lambda^V$ for all $x \in H_\lambda^V$ of size less than

⁹Here, a poset is *nontrivial* if it necessarily adds a new set.

δ in V . It follows from the δ -approximation property of $V \subseteq V'$ that $X \in V$. Finally, X has size less than λ in V' , but then it must have size less than λ in V as well, showing that $X \in H_\lambda^V$ and that $H_\lambda^V \subseteq H_\lambda^{V'}$ satisfies the δ -approximation property. \square

This lemma shows that we can deduce results about Ramsey and Ramsey-like ranks in class-sized extensions with cover and approximation properties from corresponding results about extensions of practical models, since we may always restrict to sufficiently large H_λ .

Theorem 4.4. *Suppose that \mathcal{M} and \mathcal{M}' are practical for κ and that $\mathcal{M} \subseteq \mathcal{M}'$ has the δ -cover and δ -approximation properties for some regular cardinal δ of \mathcal{M}' and the same ordinals. If $\kappa > \delta$ is strongly Ramsey in \mathcal{M}' , then κ was already strongly Ramsey in \mathcal{M} . The same result holds for super Ramsey cardinals.*

The proof adapts techniques developed in [Ham03] to the embeddings characterizing strongly Ramsey cardinals. We will note in the course of the argument where the constructions occurred in [Ham03].

Proof. First, suppose that $\kappa > \delta$ is strongly Ramsey in \mathcal{M}' . Fix $A \subseteq \kappa$ in \mathcal{M} . We need to show that \mathcal{M} has a κ -model M containing A and a weakly amenable M -ultrafilter U on κ . In \mathcal{M}' , let W be an $\{A, V_\kappa^{\mathcal{M}}\}$ -good strong Ramsey measure and let $j : M_W \rightarrow N_W$ be the ultrapower by W . Note that both $V_\kappa^{\mathcal{M}'}$ and $V_\kappa^{\mathcal{M}}$ are ZFC-models since κ is inaccessible in \mathcal{M}' and therefore also in \mathcal{M} .

Claim 4.4.1. *The pair $V_\kappa^{\mathcal{M}} \subseteq V_\kappa^{\mathcal{M}'}$ has the δ -cover and δ -approximation properties.*

Proof. This is just Lemma 4.3. \square

Thus, by elementarity, N_W satisfies that the pair $N = j(V_\kappa^{\mathcal{M}}) \subseteq j(V_\kappa^{\mathcal{M}'}) = N'$ has the δ -cover and δ -approximation properties, and it is correct about this. Observe that $N' = V_{j(\kappa)}^{N_W}$, and therefore is a κ -model in \mathcal{M}' since N_W is a κ -model in \mathcal{M}' (it is not difficult to see that the ultrapower of κ -model is always a κ -model). The next several claims have the aim to conclude that N and $W \cap N$ are in \mathcal{M} .

Claim 4.4.2. $V_\kappa^N = V_\kappa^{\mathcal{M}}$.

Proof. If $X \in V_\kappa^{\mathcal{M}}$, then $X = j(X) \in j(V_\kappa^{\mathcal{M}}) = N$. So $V_\kappa^{\mathcal{M}} \subseteq N$. Conversely, if $X \in V_\kappa^N$, then note first that $X \in M_W = \text{dom}(j)$. This is because $X \in V_\kappa^{N_W} = V_\kappa^{M_W}$. But then $j(X) = X \in N = j(V_\kappa^{\mathcal{M}})$, and so $X \in V_\kappa^{\mathcal{M}}$. \square

Claim 4.4.3. *If $X \subseteq \text{ORD}^N$ is a set of size less than δ in \mathcal{M}' , then there is $Y \in \mathcal{M} \cap N$ of size at most δ in N' such that $X \subseteq Y$.*

Proof. This construction mimics Lemma 3.2 in [Ham03]. Let $X_0 = X$, and observe that $X_0 \in N'$ since N' is a κ -model in \mathcal{M}' . So, by the δ -cover property of $N \subseteq N'$, there is $X_1 \subseteq \text{ORD}^N$ of size less than δ in N such that $X_0 \subseteq X_1$. Then, by the δ -cover property of $\mathcal{M} \subseteq \mathcal{M}'$, there is $X_2 \subseteq \text{ORD}^N$ of size less than δ in \mathcal{M} such that $X_1 \subseteq X_2$ (this uses that \mathcal{M} and \mathcal{M}' have the same ordinals). The set X_2 is in the κ -model N' , and so, again, there is X_3 of size less than δ in N such that $X_2 \subseteq X_3$. Continue bouncing between N and \mathcal{M} in this way. To get through limit stages, observe that if $\gamma < \delta$ and $\langle X_\xi \mid \xi < \gamma \rangle$ is a sequence of sets of size less than δ in \mathcal{M}' , then $X_\gamma = \bigcup_{\xi < \gamma} X_\xi$ has size less than δ in \mathcal{M}' by the regularity of δ . Thus, after δ -many steps, we end up with an increasing sequence $\langle X_\xi \mid \xi < \delta \rangle$

such that cofinally many elements of it are in N and cofinally many are in \mathcal{M} . Let $Y = \bigcup_{\xi < \delta} X_\xi$. By closure, $Y \in N'$ and Y has size at most δ there. To see that $Y \in N$ we use the δ -approximation property of $N \subseteq N'$. Specifically, let $y \in N$ have size less than δ . Then there is some $\xi < \delta$ such that $Y \cap y = X_\xi \cap y$ and we may furthermore choose ξ so that $X_\xi \in N$. So then clearly $Y \cap y \in N$ and we obtain $Y \in N$ by the δ -approximation property. A similar argument, using the δ -approximation property of $\mathcal{M} \subseteq \mathcal{M}'$, shows that also $Y \in \mathcal{M}'$. \square

Claim 4.4.4. \mathcal{M} and N have the same subsets of ORD^N of size less than δ in \mathcal{M}' .

Proof. This argument mimics Lemma 3.3 in [Ham03]. Suppose that $X \subseteq \text{ORD}^N$ has size less than δ in \mathcal{M}' . By Claim 4.4.3, there is a set Y of size at most δ in N' such that $X \subseteq Y$ and $Y \in N \cap \mathcal{M}$. Let $Y = \{y_\alpha \mid \alpha < \gamma\}$ be the enumeration of Y arising from its order-type and note that $\gamma < \delta^{+\mathcal{M}'} < \kappa$. Since the order-type of Y is absolute, the enumeration is in both N and \mathcal{M} . Let $\bar{X} = \{\alpha < \gamma \mid y_\alpha \in X\}$, which is a subset of γ . Now observe that X is in N or \mathcal{M} if and only if \bar{X} is there, and \bar{X} is in \mathcal{M} if and only if it is in N , by Claim 4.4.2, since it is a subset of γ . \square

Claim 4.4.5. $\mathcal{M} \cap N' = N$ and $N \in \mathcal{M}$.

Proof. This argument mimics Lemma 3.4 in [Ham03]. First, we show that $N \subseteq \mathcal{M}$. It suffices to verify that all sets of ordinals in N are elements of \mathcal{M} . Suppose that $X \subseteq \text{ORD}^N$ is in N . Fix a set $x \subseteq \text{ORD}^N$ of size less than δ in \mathcal{M} . By Claim 4.4.4 we have $x \in N$ and so $X \cap x \in N$ as well. But $X \cap x$ is a set of ordinals in N of size less than δ in \mathcal{M}' , and therefore by Claim 4.4.4 again, $X \cap x \in \mathcal{M}$. So $X \in \mathcal{M}$ by the δ -approximation property of $\mathcal{M} \subseteq \mathcal{M}'$.

Next, we verify that $\mathcal{M} \cap N' \subseteq N$. Initially, we show that every set of ordinals in $\mathcal{M} \cap N'$ is in N . So suppose that X is a set of ordinals in $\mathcal{M} \cap N'$. Let x be a set of ordinals of size less than δ in N . Then $x \in \mathcal{M}$, by Claim 4.4.4, and so $x \cap X$ in \mathcal{M} . By Claim 4.4.4 again, $x \cap X \in N$. So, by the δ -approximation property of $N \subseteq N'$, we obtain $X \in N$. Now suppose that X is any set in $\mathcal{M} \cap N'$. By \in -induction, suppose that every element of X is in N . Since $X \in N'$ and N' is a model of ZFC, there must be some ordinal β in N' such that $X \in V_\beta^{N'}$ and thus $X \subseteq V_\beta^N = Y$. Enumerate $Y = \{y_\alpha \mid \alpha < \gamma\}$ in N , and note that since $N \subseteq \mathcal{M}$, the enumeration exists in \mathcal{M} as well. Let $\bar{Y} = \{\alpha < \gamma \mid y_\alpha \in X\}$. The set \bar{Y} is in \mathcal{M} and also in N' . So by what we already argued for sets of ordinals, \bar{Y} is in N , and hence so is X . This completes the argument that $N = \mathcal{M} \cap N'$.

We will use the δ -approximation property of $\mathcal{M} \subseteq \mathcal{M}'$ to argue that $N \in \mathcal{M}$. Fix a set x of size less than δ in \mathcal{M} . The intersection $N \cap x$ is in the κ -model N' , therefore there is some β such that $N \cap x \in V_\beta^{N'}$. It now follows, since obviously $N \cap x \subseteq N$, that $N \cap x \subseteq V_\beta^N$. Hence $N \cap x = V_\beta^N \cap x$ is in \mathcal{M} since both x and V_β^N are there. \square

Let $\bar{U} = N \cap W$.

Claim 4.4.6. $\bar{U} \in \mathcal{M}$.

Proof. We will use the δ -approximation property of $\mathcal{M} \subseteq \mathcal{M}'$. This construction mimics Theorem 10 in [Ham03]. Suppose that x is a set of size less than δ in \mathcal{M} . We can assume that $x \subseteq P(\kappa)^N$ and also that whenever some $B \subseteq \kappa$ is in x , then so is the complement of B in κ . Since W is an M_W -ultrafilter and M_W is a

κ -model in \mathcal{M}' , it follows that W is κ -intersecting in \mathcal{M}' . So working in \mathcal{M}' , we consider the intersection of all $B \in W \cap x$, which is non-empty, and hence must contain some element β . We will argue that, for $B \in x$, we have $\beta \in B$ precisely when $B \in W$. By definition of β , if $B \in W \cap x$, then $\beta \in B$. If $B \notin W$, then its complement B^c is in $x \cap W$, and so $\beta \in B^c$, which means that $\beta \notin B$. Thus, $\bar{U} \cap x = W \cap x = \{B \in x \mid \beta \in B\}$, which is clearly in \mathcal{M} . \square

Claim 4.4.7. *N is closed under $<\kappa$ -sequences in \mathcal{M} , we have $A \in N$, and \bar{U} is a weakly amenable ω_1 -intersecting N -ultrafilter.*

Proof. Since $A \in N' = V_{j(\kappa)}^{N_W}$ and also $A \in \mathcal{M}$, we get $A \in N$ by Claim 4.4.5. If $\vec{x} = \langle x_\xi \mid \xi < \gamma \rangle$, for some $\gamma < \kappa$, is a sequence of elements of N in \mathcal{M} , then \vec{x} is in both \mathcal{M} and N' , and so $\vec{x} \in N$, again by Claim 4.4.5.

It is clear that \bar{U} is an ω_1 -intersecting N -ultrafilter. It remains to show that \bar{U} is weakly amenable to N . Consider $X \cap \bar{U}$, where X is a set of size κ in N . The set $X \cap \bar{U}$ is in \mathcal{M} and also in N' by the weak amenability of U . Hence $X \in N$ by Claim 4.4.5. \square

We return to the proof of the theorem. We now have, in \mathcal{M} , a model N , closed under $<\kappa$ -sequences, with $A \in N$ and a weakly amenable ω_1 -intersecting N -ultrafilter. The only obstacle to seeing that κ is strongly Ramsey in \mathcal{M} is that N might be too large in cardinality. But this is easily fixed by building an elementary substructure M of $H_{\kappa^+}^N$ containing A in κ -many steps so that M is a κ -model and $U = \bar{U} \cap M$ is a weakly amenable M -ultrafilter.

To prove the same result for super Ramsey cardinals, we start with $M_W \prec H_{\kappa^+}^{\mathcal{M}'}$. We will be done if we can argue that $H_{\kappa^+}^N \prec H_{\kappa^+}^{\mathcal{M}'}$. First, observe that we can use the δ -approximation property of $\mathcal{M} \subseteq \mathcal{M}'$ to define $\mathcal{P}(\kappa)^{\mathcal{M}}$ in $H_{\kappa^+}^{\mathcal{M}'}$, using the parameter $a = V_\kappa^{\mathcal{M}}$, as the collection of all $X \subseteq \kappa$ such that for all $x \in a$, $x \cap X \in a$. Let's see that the same formula defines $\mathcal{P}(\kappa)^N$ in $H_{\kappa^+}^{N'}$. Recall that $a = V_\kappa^N \in N$ by Claim 4.4.1. If $X \subseteq \kappa$ is in N , then $x \cap X \in a$ for every $x \in a$, and if $X \subseteq \kappa$ is in N' and $x \cap X \in a$ for all $x \in a$, then $X \in N$ by the δ -approximation property of $N \subseteq N'$. Thus, using the usual Mostowski coding, there is a formula $\psi(x, a)$ which defines $H_{\kappa^+}^{\mathcal{M}}$ in $H_{\kappa^+}^{\mathcal{M}'}$ and the same $\psi(x, a)$ also defines $H_{\kappa^+}^N$ in $H_{\kappa^+}^{N'}$. Now suppose that $H_{\kappa^+}^{\mathcal{M}} \models \exists x \varphi(x, b)$ for some $b \in H_{\kappa^+}^N$. So $H_{\kappa^+}^{\mathcal{M}'}$ satisfies that $\exists x \varphi(x, b)$ holds in the collection defined by $\psi(x, a)$. Since $H_{\kappa^+}^{N'} = M_W \prec H_{\kappa^+}^{\mathcal{M}'}$, it satisfies the same statement, which gives that $H_{\kappa^+}^N \models \exists x \varphi(x, b)$. \square

Next, we show that extensions with cover and approximation properties cannot increase the rank of a strongly Ramsey or super Ramsey cardinal.

Theorem 4.5. *Suppose that \mathcal{M} and \mathcal{M}' are practical for κ and that $\mathcal{M} \subseteq \mathcal{M}'$ has the δ -cover and δ -approximation properties for some regular cardinal δ of \mathcal{M}' and the same ordinals. If $\kappa > \delta$, then $o_{\text{stRam}}(\kappa)^{\mathcal{M}'} \leq o_{\text{stRam}}(\kappa)^{\mathcal{M}}$. The same result holds for super Ramsey cardinals.*

Proof. We will argue by induction on α that if \mathcal{M} and \mathcal{M}' are as in the hypothesis and $o_{\text{stRam}}(\kappa) = \alpha$ in \mathcal{M}' , then $o_{\text{stRam}}(\kappa) \geq \alpha$ in \mathcal{M} . So suppose inductively that the statement holds for all $\beta < \alpha$. Fix some pair \mathcal{M} and \mathcal{M}' as in the hypothesis and suppose that $o_{\text{stRam}}(\kappa) = \alpha$ in \mathcal{M}' . By Corollary 3.8 we have to show that \mathcal{M} has a strong Ramsey measure U with $N_U \models o_{\text{stRam}}(\kappa) \geq \beta$ for every $\beta < \alpha$ and

$A \subseteq \kappa$ in \mathcal{M} . So fix $\beta < \alpha$ and $A \subseteq \kappa$ in \mathcal{M} . In \mathcal{M}' there is a strong Ramsey measure W such that M_W contains A and $V_\kappa^{\mathcal{M}}$ and $N_W \models o_{\text{stRam}}(\kappa) = \beta$. In what follows we use the notation from the proof of Theorem 4.4. Construct N and N' . Clearly $N' \models o_{\text{stRam}}(\kappa) = \beta$, since $N' = V_{j(\kappa)}^{N_W}$. Since $N \subseteq N'$ has the δ -cover and δ -approximation properties, we may apply the induction hypothesis to this pair and conclude that $N \models o_{\text{stRam}}(\kappa) \geq \beta$. Corollary 3.8 now implies that $H_{\kappa^+}^N$ satisfies that for every $B \subseteq \kappa$ and $\xi < \beta$ there is a strong Ramsey measure U_B with $N_{U_B} \models o_{\text{stRam}}(\kappa) \geq \xi$ and so $M \prec H_{\kappa^+}^N$ must satisfy this statement as well. It follows that the ultrapower of M by U must, by weak amenability, satisfy the same statement, meaning that $o_{\text{stRam}}(\kappa) \geq \beta$ there, which is precisely what we set out to establish. \square

4.2. Ramsey cardinals. The arguments presented in the previous section do not generalize directly to Ramsey cardinals because we can no longer work with κ -models, whose properties were used crucially in several places in the proof of Theorem 4.4 to pass between \mathcal{M}' and N' . Nevertheless, we can modify the proof to work for Ramsey cardinals with the extra assumption that \mathcal{M}' doesn't have new countable sequences of elements of \mathcal{M} .

Theorem 4.6. *Suppose that \mathcal{M} and \mathcal{M}' are practical for κ and that $\mathcal{M} \subseteq \mathcal{M}'$ has the δ -cover and δ -approximation properties for some regular cardinal δ of \mathcal{M}' , they have the same ordinals and $\mathcal{M}^\omega \subseteq \mathcal{M}$ in \mathcal{M}' . If $\kappa > \delta$ is Ramsey in \mathcal{M}' , then κ was already Ramsey in \mathcal{M} .*

Proof. Fix some $A \subseteq \kappa$ in \mathcal{M} . In \mathcal{M}' , let W be an $\{A, V_\kappa^{\mathcal{M}}\}$ -good Ramsey measure and let $j : M_W \rightarrow N_W$ be the ultrapower by W . We proceed at first as in the proof of Theorem 4.4. Let $N = j(V_\kappa^{\mathcal{M}})$ and $N' = j(V_\kappa^{\mathcal{M}'})$. As before, the pair $N \subseteq N'$ has the δ -cover and δ -approximation properties, but N' is no longer a κ -model. Also, as before, $V_\kappa^N = V_\kappa^{\mathcal{M}}$.

Claim 4.6.1. $\mathcal{P}^N(\kappa) = \mathcal{M} \cap \mathcal{P}^{N'}(\kappa)$ and hence $H_{\kappa^+}^N \subseteq \mathcal{M}$.

Proof. First, we show that $\mathcal{P}^N(\kappa) \subseteq \mathcal{M}$ using the δ -approximation property of $\mathcal{M} \subseteq \mathcal{M}'$. So suppose that $B \subseteq \kappa$ is in N . Fix a set $x \subseteq \kappa$ of size less than δ in \mathcal{M} , and note that $x \in V_\kappa^{\mathcal{M}} \subseteq N$. Thus, $x \cap B \in V_\kappa^N \subseteq \mathcal{M}$ and the δ -approximation property gives $B \in \mathcal{M}$.

Next, we show that subsets of κ in the intersection of \mathcal{M} and N' must be in N using the δ -approximation property of $N \subseteq N'$. So suppose that $B \subseteq \kappa$ and $B \in \mathcal{M} \cap N'$. Let $x \subseteq \kappa$ be a set of size less than δ in N , meaning that $x \in V_\beta^{\mathcal{M}}$ for some $\beta < \kappa$. So $x \cap B$ is also in $V_\beta^{\mathcal{M}}$, and hence is in N .

Finally, any $X \in H_{\kappa^+}^N$ is coded by a subset of κ via Mostowski coding, and this coding can be undone in \mathcal{M} . \square

We cannot prove that N or even $H_{\kappa^+}^N$ is an element of \mathcal{M} . So instead we will find a weak κ -model $M \prec H_{\kappa^+}^N$ for which $U = M \cap W$ is a weakly amenable M -ultrafilter so that both M and U are in \mathcal{M} .

First, we argue that $\overline{W} = W \cap N$ is weakly amenable to N . Let S be a subset of $\mathcal{P}(\kappa)$ of size κ in N . By weak amenability $\overline{S} = S \cap W$ is in N' . Now we will use the δ -approximation property of $N \subseteq N'$ to get \overline{S} into N . Let $x \subseteq S$ be a set of size less than δ in N . We can assume that whenever $B \subseteq \kappa$ is in x , then so is the complement of B in κ . Since W is an N_W -ultrafilter and $\overline{S} \cap x$ is in N_W , it follows

that there is some β that is an element of every $B \in \overline{S} \cap x$. Thus, the sets in $\overline{S} \cap x$ are precisely the sets in x having β as an element, and so $\overline{S} \cap x$ is in N .

Now we build $M \prec H_{\kappa^+}^N$, working in \mathcal{M}' , as in the proof of Lemma 2.1 from the sequence $\langle (M_n, U_n) \mid n < \omega \rangle$, so that $M = \bigcup_{n < \omega} M_n$ and $U = \bigcup_{n < \omega} U_n \subseteq W$ is a weakly amenable ω_1 -intersecting M -ultrafilter. Since each M_n and U_n are in $H_{\kappa^+}^N \subseteq \mathcal{M}$, it follows by our closure assumption that M and U are in \mathcal{M} . \square

Theorem 4.7. *Suppose that \mathcal{M} and \mathcal{M}' are practical for κ and that $\mathcal{M} \subseteq \mathcal{M}'$ has the δ -cover and δ -approximation properties for some regular cardinal δ of \mathcal{M}' , they have the same ordinals and $\mathcal{M}^\omega \subseteq \mathcal{M}$ in \mathcal{M}' . If $\kappa > \delta$, then $o_{\text{Ram}}(\kappa)^{\mathcal{M}'} \leq o_{\text{Ram}}(\kappa)^{\mathcal{M}}$.*

The proof is identical to Theorem 4.5.

Question 4.8. Can we remove the assumption that $\mathcal{M}^\omega \subseteq \mathcal{M}$ in \mathcal{M}' from Theorem 4.7?

4.3. α -iterable cardinals. For completeness, we will argue here that extensions with cover and approximation properties cannot create new α -iterable cardinals provided that the extension has no new countable sequences from the old model.

Theorem 4.9. *Suppose that \mathcal{M} and \mathcal{M}' are practical for κ and that $\mathcal{M} \subseteq \mathcal{M}'$ has the δ -cover and δ -approximation properties for some regular cardinal δ of \mathcal{M}' , they have the same ordinals and $\mathcal{M}^\omega \subseteq \mathcal{M}$ in \mathcal{M}' . If $\kappa > \delta$ is α -iterable in \mathcal{M}' , then it was already α -iterable in \mathcal{M} .*

Proof. Fix $A \subseteq \kappa$ in V . Working in V' , find an $\{A, V_\kappa^{\mathcal{M}}, V_\kappa^{\mathcal{M}'}\}$ -good weak κ -model M_W for which there is an α -iterable M_W -ultrafilter W . We follow the proof of Theorem 4.6 exactly by considering the pair $N \subseteq N'$. We can show that $\mathcal{P}^N(\kappa) = \mathcal{M} \cap \mathcal{P}^{N'}(\kappa)$ and that $\overline{W} = W \cap N$ is weakly amenable to N exactly as there. Working in \mathcal{M}' we then build the sequence $\langle (M_n, U_n) \mid n < \omega \rangle$ of elementary submodels of $H_{\kappa^+}^N$ and filters, and this sequence must be in V by our assumption. Thus, $M = \bigcup_{n < \omega} M_n$ and $U = \bigcup_{n < \omega} U_n$ are in V . Finally, Lemma 3.8 of [GW11] implies that U is α -iterable. \square

Question 4.10. Can we remove the assumption that $\mathcal{M}^\omega \subseteq \mathcal{M}$ in \mathcal{M}' from Theorem 4.9?

5. KILLING THE M-RANK SOFTLY

We can use forcing to softly kill the rank of a Ramsey or Ramsey-like cardinal, meaning that, if κ has rank α and $\beta < \alpha$, then there is a cofinality preserving forcing extension in which κ has rank β . Let's consider the case of Ramsey cardinals. We will obtain the desired forcing extension by carefully adding a club through ordinals δ with $o_{\text{Ram}}(\delta) < g^E(\delta)$ (where g^E is a representing function for α), while preserving $o_{\text{Ram}}(\kappa) \geq \beta$. The result will follow because no weak κ -model containing such a club can have its ultrapower satisfy $o_{\text{Ram}}(\kappa) = \alpha$.

Recall that, if U is a Ramsey or Ramsey-like measure on M_U with the ultrapower map $j: M_U \rightarrow N_U$, then $M_U^* = V_{j(\kappa)}^{N_U}$.

Lemma 5.1. *Suppose $o_{\text{Ram}}(\kappa) = \alpha > 0$. Then for every $A \subseteq \kappa$ and $\beta < \alpha$, there is an A -good Ramsey measure U such that M_U is ω -special and $N_U \models o_{\text{Ram}}(\kappa) = \beta$, and hence M_U^* is ω -special and $M_U^* \models o_{\text{Ram}}(\kappa) = \beta$.*

Proof. Fix $A \subseteq \kappa$ and let \bar{U} be any A -good Ramsey measure with $N_{\bar{U}} \models o_{\text{Ram}}(\kappa) = \beta$. Following the proof of Lemma 2.1, we construct, in ω -many steps, an A -good Ramsey measure U such that $M_U \prec M_{\bar{U}}$ is ω -special and $\beta \in M_U$. By Corollary 3.8 $M_{\bar{U}}$ has a B -good Ramsey measure W with $N_W \models o_{\text{Ram}}(\kappa) \geq \xi$ for every $\xi < \beta$ and $B \subseteq \kappa$ in $M_{\bar{U}}$ and β is the largest ordinal for which this is true. Thus, by elementarity, M_U has a B -good Ramsey measure W with $N_W \models o_{\text{Ram}}(\kappa) \geq \xi$ for every $\xi < \beta$ and $B \subseteq \kappa$ in M_U and β is still the largest ordinal for which this is true. It follows that $N_U \models o_{\text{Ram}}(\kappa) = \beta$.

Recall from Section 2 that whenever M_U is ω -special, then so is M_U^* . Since $M_U^* = V_{j(\kappa)}^{N_U}$, where j is the ultrapower map by U , it satisfies $o_{\text{Ram}}(\kappa) = \beta$. \square

Theorem 5.2. *If $o_{\text{Ram}}(\kappa) = \alpha$ and $\beta < \alpha$ is any ordinal, then there is a cofinality preserving forcing extension in which $o_{\text{Ram}}(\kappa) = \beta$. The same result holds for strongly Ramsey and super Ramsey cardinals.*

Proof. Suppose $o_{\text{Ram}}(\kappa) = \alpha$ and fix $\beta < \alpha$. Since $\beta < \kappa^+$, we can fix some well-ordering E of κ in order-type β and let $g^E : \kappa \rightarrow \kappa$ be a representing function for β (see the discussion preceding Theorem 3.7; if $\beta < \kappa$ we can let β be represented by a constant function and omit E and g^E from the following argument).

Let \mathbb{P}_κ be the κ -length Easton support iteration, forcing at each inaccessible γ with \mathbb{Q}_γ to shoot a club, by closed initial segments, through the set of cardinals $\delta < \gamma$ with $o_{\text{Ram}}(\delta) < g^E(\delta)$, and using trivial forcing everywhere else. It is easy to see that each \mathbb{Q}_γ is $< \gamma$ -strategically closed. Fixing $\beta < \gamma$, the strategy to ensure that the union of a β -sequence of conditions in \mathbb{Q}_γ with the supremum added on is itself a condition in \mathbb{Q}_γ is to make sure that the supremum gets above β . This ensures that the supremum is not inaccessible and so trivially has the property $o_{\text{Ram}}(\delta) < g^E(\delta)$. The forcing we shall use to achieve our goal is $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$. This poset preserves all cardinals and cofinalities, since each \mathbb{Q}_γ is $< \gamma$ -strategically closed in $V^{\mathbb{P}_\gamma}$. Let $G * g \subseteq \mathbb{P}$ be V -generic.

The iteration \mathbb{P}_κ has size κ and the κ -cc (cf. [Cum10]) and elements of $\dot{\mathbb{Q}}_\kappa$ are names for bounded subsets of κ . Since each such name can be associated with a bounded subset of κ by a nice-name argument, we can assume that $\mathbb{P} \subseteq V_\kappa$. This means in particular that every $A \subseteq \kappa$ in $V[G][g]$ has a \mathbb{P} -name \dot{A} in H_{κ^+} and so A is an element of every model $M[G][g]$ where M is a weak κ -model in V containing \mathbb{P} and \dot{A} . The following claim will show that the rank of κ in $V[G][g]$ is still at least β .

Claim 5.2.1. *If $\mathcal{M} \models \text{ZFC}$ is a weak κ -model such that $V_\kappa, \beta, E \in \mathcal{M}$ and $o_{\text{Ram}}(\kappa)^\mathcal{M} < \beta$, then $o_{\text{Ram}}(\kappa)^{\mathcal{M}[G][g]} \geq o_{\text{Ram}}(\kappa)^\mathcal{M}$.*

Proof. Note that $\mathbb{P} \in \mathcal{M}$ since it is definable from V_κ and E . We shall argue by induction on $\xi < \beta$ that if \mathcal{M} is as in the hypothesis and $o_{\text{Ram}}(\kappa)^\mathcal{M} = \xi$, then $o_{\text{Ram}}(\kappa)^{\mathcal{M}[G][g]} \geq \xi$. So suppose inductively that the statement holds for all $\eta < \xi$. Fix some \mathcal{M} as in the hypothesis and suppose that $o_{\text{Ram}}(\kappa)^\mathcal{M} = \xi$. For $A \subseteq \kappa$ in $\mathcal{M}[G][g]$ and $\eta < \xi$, we need to produce an A -good Ramsey measure W such that $N_W \models o_{\text{Ram}}(\kappa) \geq \eta$.

Let $A \in \mathcal{M}[G][g]$ be a subset of κ and choose a \mathbb{P} -name $\dot{A} \in \mathcal{M}$ for it. Fix $\eta < \xi$. We work in \mathcal{M} . By Lemma 5.1 we can find an $\{\dot{A}, V_\kappa, \beta, E\}$ -good Ramsey measure U such that M_U is ω -special and $N_U \models o_{\text{Ram}}(\kappa) = \eta$. Let $M = M_U^*$ and $h : M \rightarrow N$ be the ultrapower by U . Note that $M \models o_{\text{Ram}}(\kappa) = \eta$, and so the

inductive assumption applied to M gives that $M[G][g] \models o_{\text{Ram}}(\kappa) \geq \eta$. We shall lift h to $M[G][g]$ in $\mathcal{M}[G][g]$ and argue that the $M[G][g]$ -ultrafilter by which the lift is the ultrapower is the desired W .

First, we lift h to $M[G]$. To do this we need to find an N -generic filter for $h(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa * \dot{\mathbb{P}}_{\text{tail}}$ containing $h \restriction G = G$. We will use the filter $G * g$ for the $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ part of $h(\mathbb{P}_\kappa)$. Note that $\mathbb{P}_{\text{tail}} = (\dot{\mathbb{P}}_{\text{tail}})_{G * g}$ is $\leq \kappa$ -strategically closed and hence $\leq \kappa$ -distributive in $N[G][g]$. Thus, by Lemma 2.2, $M[G][g]$ has an $N[G][g]$ -generic for G_{tail} for \mathbb{P}_{tail} , and so we can lift h to $h : M[G] \rightarrow N[h(G)]$, where $h(G) = G * g * G_{\text{tail}}$.

Next, we lift h fully to $M[G][g]$ by finding an $N[h(G)]$ -generic filter for $h(\mathbb{Q}_\kappa)$, where $\mathbb{Q}_\kappa = (\dot{\mathbb{Q}}_\kappa)_G$, containing $h \restriction g$. Let $C = \bigcup g$ and $\bar{C} = C \cup \{\kappa\}$, which is in $N[h(G)]$ by our choice of $h(G)$. Note that \bar{C} is a closed bounded subset of $h(\kappa)$. If we can show that \bar{C} is a condition in $h(\mathbb{Q}_\kappa)$, then we can use it as a master condition for the lift and use Lemma 2.2 to find an $N[h(G)]$ -generic filter g^* for $h(\mathbb{Q}_\kappa)$ containing \bar{C} . The only reason it might not be the case that \bar{C} is an element of \mathbb{Q}_κ is that $o_{\text{Ram}}(\delta)^{N[h(G)]} \geq g^E(\delta)$ for some $\delta \in C$ or $o_{\text{Ram}}(\kappa)^{N[h(G)]} \geq h(g^E)(\kappa) = \beta$.

The first option cannot occur, since otherwise we would get $o_{\text{Ram}}(\delta)^{M[G]} \geq g^E(\delta)$ by elementarity. This contradicts the construction of C which is a club of ordinals δ satisfying $o_{\text{Ram}}(\delta)^{M[G]} < g^E(\delta)$.

To see that the second option above also cannot occur, observe that the forcing $h(\mathbb{P}_\kappa)$ has a closure point at the first inaccessible cardinal δ_0 : the first non-trivial forcing happens at stage δ_0 and has size δ_0 and each $\dot{\mathbb{Q}}_\delta$ for $\delta > \delta_0$ is $\leq \delta_0$ -strategically closed in $V^{\mathbb{P}^\delta}$, from which it will follow that the remainder of the iteration is $\leq \delta_0$ -strategically closed. By Theorem 4.2 the pair $N \subseteq N[h(G)]$ has the δ_0^+ -cover and δ_0^+ -approximation properties. Also $h(\mathbb{P}_\kappa)$ is clearly countably closed, meaning that $N^\omega \subseteq N$ in $N[h(G)]$. Following our assumptions, $N \models o_{\text{Ram}}(\kappa) = \eta$ and hence Theorem 4.7 yields that $N[h(G)] \models o_{\text{Ram}}(\kappa) \leq \eta < \beta$. This completes the argument that \bar{C} is a condition in $h(\mathbb{Q}_\kappa)$, allowing us to lift h to $h : M[G][g] \rightarrow N[h(G)][g^*]$, where g^* is obtained using Lemma 2.2 below the master condition \bar{C} .

Next, we argue that the lift h is the ultrapower by a weakly amenable ω_1 -intersecting $M[G][g]$ -ultrafilter W in $\mathcal{M}[G][g]$. By Lemma 2.3, since \mathbb{P} is countably closed, W is ω_1 -intersecting. To conclude that W is weakly amenable, we verify that $M[G][g]$ and $N[h(G)][g^*]$ have the same subsets of κ . Suppose B is a subset of κ in $N[h(G)][g^*]$. Since $\mathbb{P}_{\text{tail}} * h(\dot{\mathbb{Q}}_\kappa)$ is $\leq \kappa$ -distributive, $B \in N[G][g]$ and so B has a \mathbb{P} -name $\dot{B} \in N$, which we can take to be an element of $H_{\kappa^+}^N = M$. So finally, $B \in M[G][g]$.

Recall that $M[G][g] \models o_{\text{Ram}}(\kappa) \geq \eta$, from which it follows that $N_W \models o_{\text{Ram}}(\kappa) \geq \eta$ as well. This finishes the inductive argument and allows us to conclude that $\mathcal{M}[G][g] \models o_{\text{Ram}}(\kappa) \geq \xi$. \square

To see that $o_{\text{Ram}}(\kappa)^{V[G][g]} \leq \beta$, recall that $C = \bigcup g$ is a club in κ , consisting of cardinals δ with $o_{\text{Ram}}(\delta)^{V[G]} < g^E(\delta)$. But since \mathbb{Q}_κ is $< \kappa$ -distributive, it also follows that $o_{\text{Ram}}(\delta)^{V[G][g]} < g^E(\delta)$ for all $\delta \in C$. This means that in $V[G][g]$ there cannot be a Ramsey measure U with $C, E \in M_U$ and $N_U \models o_{\text{Ram}}(\kappa) \geq j(g^E)(\kappa) = \beta$.

Exactly the same argument would work to get the result for strongly and super Ramsey cardinals, except that we would rely on Theorem 4.5. \square

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