A MODEL OF THE GENERIC VOPĚNKA PRINCIPLE IN WHICH THE ORDINALS ARE NOT Δ_2-MAHLO

VICTORIA GITMAN AND JOEL DAVID HAMKINS

ABSTRACT. The generic Vopěnka principle, we prove, is relatively consistent with the ordinals being non-Mahlo. Similarly, the generic Vopěnka scheme is relatively consistent with the ordinals being definably non-Mahlo. Indeed, the generic Vopěnka scheme is relatively consistent with the existence of a Δ_2-definable class containing no regular cardinals. In such a model, there can be no Σ_2-reflecting cardinals and hence also no remarkable cardinals. This latter fact answers negatively a question of Bagaria, Gitman and Schindler.

1. INTRODUCTION

The *Vopěnka principle* is the assertion that for every proper class of first-order structures in a fixed language, one of the structures embeds elementarily into another. This principle can be formalized as a single second-order statement in Gödel-Bernays set-theory GBC, and it has a variety of useful equivalent characterizations. For example, the Vopěnka principle holds precisely when for every class A, the universe has an A-extendible cardinal, and it is also equivalent to the assertion that for every class A, there is a stationary proper class of A-extendible cardinals (see [Ham, theorem 6]). In particular, the Vopěnka principle implies that ORD is Mahlo: every class club contains a regular cardinal and indeed, an extendible cardinal and more.

To define these terms, recall that a cardinal κ is *extendible*, if for every $\lambda > \kappa$, there is an ordinal θ and an elementary embedding $j : V_\lambda \to V_\theta$ with critical point κ. It turns out that, in light of the Kunen inconsistency, this weak form of extendibility is equivalent to a stronger form, where one insists also that $\lambda < j(\kappa)$; but there is a subtle issue about this that will come up later in our treatment of the virtual forms of these axioms, where the virtual weak and virtual strong forms are no longer equivalent. Relativizing to a class parameter, a cardinal κ is *A-extendible* for a class A, if for every $\lambda > \kappa$, there is an elementary embedding

$$j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \to \langle V_\theta, \in, A \cap V_\theta \rangle$$

with critical point κ, and again one may equivalently insist also that $\lambda < j(\kappa)$. Every such A-extendible cardinal is therefore extendible and hence inaccessible, measurable, supercompact and more. These are amongst the largest large cardinals.

In the first-order ZFC context, set theorists commonly consider a first-order version of the Vopěnka principle, which we call the *Vopěnka scheme*, the scheme making the Vopěnka assertion of each definable class separately, allowing parameters.[1] That is, the Vopěnka scheme asserts, of every formula φ, that for any parameter

---

[1]Henceforth in this article, when we say 'definable class,' we shall always mean that parameters are allowed, unless otherwise specifically mentioned.

p, if $\{\, x \mid \varphi(x,p) \,\}$ is a proper class of first-order structures in a common language, then one of those structures elementarily embeds into another.

The Vopěnka scheme is naturally stratified by the assertions $\mathrm{VP}(\Sigma_n)$, for the particular natural numbers n in the meta-theory, where $\mathrm{VP}(\Sigma_n)$ makes the Vopěnka assertion for all Σ_n-definable classes. Using the definable Σ_n-truth predicate, each assertion $\mathrm{VP}(\Sigma_n)$ can be expressed as a single first-order statement in the language of set theory.

Hamkins [Ham] proved that the Vopěnka principle is not provably equivalent to the Vopěnka scheme, if consistent, although they are equiconsistent over GBC and furthermore, the Vopěnka principle is conservative over the Vopěnka scheme for first-order assertions. That is, over GBC the two versions of the Vopěnka principle have exactly the same consequences in the first-order language of set theory.

In this article, we are concerned with the virtual forms of the Vopěnka principles. The main idea of virtualization, due to Schindler, is to weaken elementary-embedding existence assertions to the assertion that such embeddings can be found in a forcing extension of the universe. Gitman and Schindler [GS] emphasized that the remarkable cardinals, for example, instantiate the virtualized form of supercompactness via the Magidor characterization of supercompactness. This virtualization program has now been undertaken with various large cardinals, leading to fruitful new insights (see [GS, BGS17]).

Carrying out the virtualization idea with the Vopěnka principles, we define the *generic Vopěnka principle* to be the second-order assertion in GBC that for every proper class of first-order structures in a common language, one of the structures admits, in some forcing extension of the universe, an elementary embedding into another. That is, the structures themselves are in the class in the ground model, but you may have to go to the forcing extension in order to find the elementary embedding.

Similarly, the *generic Vopěnka scheme*, introduced in [BGS17], is the assertion (in ZFC or GBC) that for every first-order definable proper class of first-order structures in a common language, one of the structures admits, in some forcing extension, an elementary embedding into another.

On the basis of their work in [BGS17], Bagaria, Gitman and Schindler had asked the following question:

Question 1. If the generic Vopěnka scheme holds, then must there be a proper class of remarkable cardinals?

There seemed good reason to expect an affirmative answer, even assuming only $\mathrm{gVP}(\Sigma_2)$, based on strong analogies with the non-generic case. Specifically, in the non-generic context Bagaria had proved that $\mathrm{VP}(\Sigma_2)$ was equivalent to the existence of a proper class of supercompact cardinals, while in the virtual context, Bagaria, Gitman and Schindler proved that the generic form $\mathrm{gVP}(\Sigma_2)$ was equiconsistent with a proper class of remarkable cardinals, the virtual form of supercompactness. Similarly, higher up, in the non-generic context Bagaria had proved that $\mathrm{VP}(\Sigma_{n+2})$ is equivalent to the existence of a proper class of $C^{(n)}$-extendible cardinals, while in the virtual context, Bagaria, Gitman and Schindler proved that the generic form $\mathrm{gVP}(\Sigma_{n+2})$ is equiconsistent with a proper class of virtually $C^{(n)}$-extendible cardinals.

But further, they achieved direct implications, with an interesting bifurcation feature that specifically suggested an affirmative answer to question 1. Namely,

what they showed at the Σ_2-level is that if there is a proper class of remarkable cardinals, then $\mathrm{gVP}(\Sigma_2)$ holds, and conversely if $\mathrm{gVP}(\Sigma_2)$ holds, then there is either a proper class of remarkable cardinals or a proper class of virtually rank-into-rank cardinals. And similarly, higher up, if there is a proper class of virtually $C^{(n)}$-extendible cardinals, then $\mathrm{gVP}(\Sigma_{n+2})$ holds, and conversely, if $\mathrm{gVP}(\Sigma_{n+2})$ holds, then either there is a proper class of virtually $C^{(n)}$-extendible cardinals or there is a proper class of virtually rank-into-rank cardinals. So in each case, the converse direction achieves a disjunction with the target cardinal and the virtually rank-into-rank cardinals. But since the consistency strength of the virtually rank-into-rank cardinals is strictly stronger than the generic Vopěnka principle itself, one can conclude on consistency-strength grounds that it isn't always relevant, and for this reason, it seemed natural to inquire whether this second possibility in the bifurcation could simply be removed. That is, it seemed natural to expect an affirmative answer to question 1, even assuming only $\mathrm{gVP}(\Sigma_2)$, since such an answer would resolve the bifurcation issue and make a tighter analogy with the corresponding results in the non-generic/non-virtual case.

In this article, however, we shall answer the question negatively. The details of our argument seem to suggest that a robust analogy with the non-generic/non-virtual principles is achieved not with the virtual $C^{(n)}$-cardinals, but with a weakening of that property that drops the requirement that $\lambda < j(\kappa)$, as explained in theorem 9. Indeed, theorem 9 seems to offer an illuminating resolution of the bifurcation aspect of the results we mentioned from [BGS17], because it provides outright virtual large-cardinal equivalents of the stratified generic Vopěnka principles. Because the resulting virtual large cardinals are not necessarily remarkable, however, our main theorem shows that it is relatively consistent with even the full generic Vopěnka principle that there are no Σ_2-reflecting cardinals and therefore no remarkable cardinals.

Main Theorem.

(1) *It is relatively consistent that* GBC *and the generic Vopěnka principle holds, yet* ORD *is not Mahlo.*

(2) *It is relatively consistent that* ZFC *and the generic Vopěnka scheme holds, yet* ORD *is not definably Mahlo, and not even* Δ_2-*Mahlo. In such a model, there can be no* Σ_2-*reflecting cardinals and therefore also no remarkable cardinals.*

These theorems are proved as theorems 11 and 12. The theorems are proved under the assumption that $0^\sharp$ exists, although this assumption can be weakened.

2. Virtual embeddings

In our main result, we shall make use of some absoluteness properties concerning virtual embeddings, and so let us review those ideas now. The following folklore result is central.

Lemma 2 (Absoluteness lemma)**.** *Suppose that M is a countable first-order structure and $j : M \to N$ is an elementary embedding. If W is a transitive (set or class) model of (some sufficiently large fragment of) ZFC such that M is countable in W and $N \in W$, then for any finite subset of M, the model W has an elementary embedding $j^* : M \to N$, which agrees with j on that subset. Moreover, if both M*

and N are transitive $\in$-structures and j has a critical point, we can additionally assume that $crit(j^*) = crit(j)$.

The proof is an elementary tree-of-attempts argument and can be found in [GS]. As a consequence, we can say a little something more about which kind of forcing extensions one needs to look in to find the embeddings: if there is an elementary embedding $j : M \to N$ in some forcing extension, then there is one in any forcing extension in which M has become countable.

Lemma 3. *If M are N are first-order structures in a common language and there is an elementary embedding $j : M \to N$ in some set-forcing extension, then there is such an embedding $j^* : M \to N$ in any forcing extension in which M has become countable. Further, one can arrange that j^* agrees with j on any prescribed finite set of values and that, if appropriate, j and j^* have the same critical point.*

To prove the lemma, note that if a forcing extension $V[G]$ has an elementary embedding $j : M \to N$, then we can go to a further extension $V[G][H]$ by any forcing collapsing M to become countable, and apply the absoluteness lemma in that extension, with the class W being the desired extension of V in which M has become countable. So there is such a j^* in that extension with the required similarities to j.

There is also an interesting game-theoretic characterization of when virtual embeddings exist, which makes no reference to forcing. Specifically, given first-order structures M and N in a common language, we define the associated two-player game $G(M, N)$ in which player I plays elements from M and player II plays corresponding elements from N. Player II wins if at every stage of play, the moves constitute a finite partial isomorphism of M to N. In other words, the type of the first n moves of player I in M is equal to the type of the first n moves of player II in N. This game $G(M, N)$ is closed for player II and therefore determined by the Gale-Stewart theorem. So one of the players has a winning strategy. The characterization is provided by the following lemma.

Lemma 4 ([BGS17])**.** *Suppose M and N are first-order structures. The following are equivalent.*

  (1) *In some set-forcing extension there is an elementary embedding $j : M \to N$.*
  (2) *Player II has a winning strategy in the game $G(M, N)$.*

One can prove this lemma simply by observing that the winner of any open game is absolute, since the recursive definition of the ordinal game values for the various positions in the game tree is defined identically in the two models. Since player II can clearly win in the forcing extension, where there is an actual elementary embedding, it follows that she must also have a winning strategy in the ground model.

Let us illustrate the easy power of these absoluteness results with the following application.

Observation 5. *Assume $0^\sharp$ exists. Then:*

  (1) *The constructible universe L, equipped with only its definable classes, is a model of the generic Vopěnka principle.*
  (2) *In L there are numerous virtual rank-into-rank embeddings $j : V_\theta^L \to V_\theta^L$, where θ is far above the supremum of the critical sequence.*

Proof. For statement (1), consider any class A that is definable (from parameters) in L. Let $j : L \to L$ be a nontrivial indiscernibility embedding that fixes the parameters, and hence which takes A to A. If κ is the critical point of j, then for any θ above κ that is closed under j, it follows that $j \restriction L_\theta : \langle L_\theta, \in, A \cap L_\theta \rangle \to \langle L_\theta, \in, A \cap L_\theta \rangle$ is an elementary embedding with critical point κ. It follows by the previous results that there is a virtual embedding like that in a forcing extension of L, and so κ is virtually A-extendible in L. Thus, the generic Vopěnka principle holds there.

For statement (2), similarly, let $j : L \to L$ shift a sequence of Silver indiscernibles down low and fix all other Silver indiscernibles. If $\lambda < \theta$ are Silver indiscernibles above the critical sequence, then $j \restriction L_\theta : L_\theta \to L_\theta$ is an elementary embedding in V with fixed point λ above the critical sequence. By the absoluteness lemma, therefore, there must be such embeddings in a forcing extension of L. □

The proof shows that every Silver indiscernible is virtually A-extendible in L for every definable class A, and furthermore, is the critical point of virtual rank-into-rank embeddings with targets as high as desired and fixed points as high above the critical sequence as desired.

One doesn't need the full strength of $0^\sharp$, however, to get a model of the generic Vopěnka principle or to get virtual rank-into-rank embeddings, and the argument above shows that $0^\sharp$ has a strictly higher consistency strength than a virtual rank-into-rank cardinal, since one can simply chop off the universe at any Silver indiscernible and reflect these assertions, gaining a transitive model of the latter theories. So the consistency strength of $0^\sharp$ is strictly stronger than necessary.

3. Large cardinal characterizations of the generic Vopěnka principle and scheme

Although our main theorem will use only the direct definition of the generic Vopěnka principle, let us sketch a richer background context for this principle by providing a large cardinal characterization of it.

Definition 6. A cardinal κ is (weakly) *virtually A-extendible*, if for every ordinal $\lambda > \kappa$ there is an ordinal θ and an elementary embedding

$$j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \to \langle V_\theta, \in, A \cap V_\theta \rangle,$$

with critical point κ.

In contrast, we define that κ is (strongly) *virtually A-extendible*, if we may also insist that $\lambda < j(\kappa)$ in the embedding mentioned above. Although in the non-virtual context mentioned in the introduction of this article, the weak and strong forms of A-extendibility coincide, nevertheless it turns out that as a consequence of the main theorem, the weak and strong forms of *virtual* A-extendibility are not the same. Furthermore, our main theorem shows that it turns out to be the weak form that is relevant for the generic Vopěnka principle, and consequently, our article is concerned principally with the weak form of virtual A-extendibility.

Theorem 7. *The generic Vopěnka principle holds if and only if for every class A, there are a proper class of (weakly) virtually A-extendible cardinals.*

Proof. We basically follow the argument of [Ham, theorem 6], which gives the non-generic/non-virtual analogue of this characterization, except that (i) the embeddings here all live in a forcing extension; (ii) there is no requirement here that

$\lambda < j(\kappa)$ for the embeddings and so this argument uses only the weak form of virtual extendibility; and (iii) we do not get here a stationary proper class of virtually A-extendible cardinals (and we cannot in light of corollary 14).

For the forward implication, assume that the generic Vopěnka principle holds and fix some class A and an ordinal γ. We will argue that there is a (weakly) virtually A-extendible cardinal above γ. We claim first that for all sufficiently large λ, there is an ordinal $\theta > \lambda$ and a virtual elementary embedding $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \to \langle V_\theta, \in, A \cap V_\theta \rangle$ with critical point above γ. If not, let $\mathcal{M}$ be the proper class of all structures $\langle V_\lambda, \in, A \cap V_\lambda, \check{\alpha} \rangle_{\alpha \leq \gamma}$, where we have added a constant symbol $\check{\alpha}$ for every ordinal $\alpha \leq \gamma$, for which there is no θ with the desired virtual embedding. Adding the constants is equivalent to requiring the critical point above γ. By the generic Vopěnka principle, there is an elementary embedding in some forcing extension between two of these structures $j : \langle V_\lambda, \in, A \cap V_\lambda, \check{\alpha} \rangle_{\alpha \leq \gamma} \to \langle V_\theta, \in, A \cap V_\theta, \check{\alpha} \rangle_{\alpha \leq \gamma}$, with $\lambda < \theta$, contrary to the inclusion of the former structure in $\mathcal{M}$, thereby establishing our claim. So we may fix an ordinal λ_0 such that for all $\lambda \geq \lambda_0$, there is an ordinal $\theta > \lambda$ and a virtual elementary embedding $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \to \langle V_\theta, \in, A \cap V_\theta \rangle$ with critical point above γ. For singular λ, we may assume without loss that j has a critical point below λ, by considering $j \restriction V_\lambda$ for an embedding j on $V_{\lambda+1}$, which must move λ, but cannot have λ as its critical point. So we have a critical point κ above γ and less than λ, although different λ could have different such critical points. Nevertheless, the map $\lambda \mapsto \kappa$, choosing the smallest such κ that can be forced to be the critical point of such an embedding, is a definable pressing-down function, and so by the class version of Fodor's lemma, there is a stationary class of λ and a cardinal κ for which we can find virtual embeddings always with critical point κ. Thus, this constant value κ is the critical point of virtual elementary embeddings $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \to \langle V_\theta, \in, A \cap V_\theta \rangle$ for unboundedly many ordinals λ. By restricting these embeddings, it follows that κ is the critical point of such virtual embeddings for every $\lambda > \kappa$, and so κ is a (weakly) virtually A-extendible cardinal above γ.

Conversely, suppose that every class A has a proper class of (weakly) virtually A-extendible cardinals, and suppose that $\mathcal{M}$ is a class of first-order structures in a common language. Let κ be a virtual $\mathcal{M}$-extendible cardinal above the size of the common language. Let λ be the κth ordinal above κ for which there is an element $M \in \mathcal{M}$ of rank λ. Since κ is virtually $\mathcal{M}$-extendible, there is an ordinal θ and a virtual elementary embedding $j : \langle V_{\lambda+1}, \in, \mathcal{M} \cap V_{\lambda+1} \rangle \to \langle V_{\theta+1}, \in, \mathcal{M} \cap V_{\theta+1} \rangle$ with critical point κ. By elementarity, it follows that $j(M) \in \mathcal{M}$ has rank $j(\lambda)$, which is the $j(\kappa)^{th}$ ordinal above $j(\kappa)$ that is the rank of an element of $\mathcal{M}$. In particular, $\lambda < j(\lambda)$ and consequently $M \neq j(M)$. Meanwhile, since the language is fixed pointwise by j, it follows that $j \restriction M : M \to j(M)$ is an elementary embedding between distinct elements of $\mathcal{M}$, thus verifying this instance of the generic Vopěnka principle. $\square$

Essentially identical arguments establish the scheme-theoretic version, where we assume that the classes A and $\mathcal{M}$ are definable.

Theorem 8. *The generic Vopěnka scheme is equivalent over* ZFC *to the scheme asserting of every definable class A that there is a proper class of virtually A-extendible cardinals.*

As we have mentioned, there is a subtle but critical difference from the corresponding results for the non-generic non-virtual forms [Ham, theorems 6,7]. Namely, in the non-generic non-virtual forms, the Vopěnka principle is equivalent both to the assertion merely that every class A has at least one A-extendible cardinal and also to the assertion that every class A has a stationary proper class of A-extendible cardinals; and similarly the Vopěnka scheme is equivalent to the corresponding assertions about definable classes A. Those implications, however, rely on the strong form of A-extendibility, and that is fine because as we have mentioned, in the non-generic context the weak and strong forms of extendibility are equivalent. In our generic/virtual context here, however, the equivalence breaks down, and the generic Vopěnka principle entitles one only to the weak form of virtual extendibility, and not the strong form. Indeed, in corollary 14 we prove that it is relatively consistent with GBC that every class A admits a proper class of weakly virtually A-extendible cardinals, but no class A admits even a single strongly virtually A-extendible cardinal.

The equivalence of the weak and strong forms of A-extendibility in the non-generic context relies fundamentally on an appeal to the Kunen inconsistency, as in [Ham, theorem 6]. In the generic/virtual case, however, there is no virtual form of the Kunen inconsistency, for one can have virtual Reinhardt cardinals, embeddings of the form $j : V_{\lambda+2} \to V_{\lambda+2}$, but where the embedding is added by forcing, as in observation 5, and this prevents one from making the analogue of the argument leading to $\lambda < j(\kappa)$ and hence from the weak form to the strong form or from individual A-extendible cardinals to stationary proper classes of them. The main result of this paper shows that this issue is inherent, since we prove that it is relatively consistent with the generic Vopěnka principle that ORD is not Mahlo. In such a model, there can be no stationary proper class of virtually extendible or virtually A-extendible cardinals, since there is not even a stationary proper class of regular cardinals.

By paying careful attention to the precise complexity of the definitions of the various classes, we may next present a stratified version of theorem 8. Let us say that a cardinal κ is (Σ_n)-*extendible*, if it is A-extendible, where A is the Σ_n-truth predicate. (Note the difference in notation from the Σ_n-extendible cardinals, used in [BHTU] with a different meaning.) Similarly, we say that κ is (weakly or strongly) *virtually (Σ_n)-extendible*, if it is (weakly or strongly) virtually A-extendible for that class.

Theorem 9. *The following are equivalent as schemes over* ZFC.
  (1) *The generic Vopěnka scheme holds for Π_{n+1}-definable classes.*
  (2) *The generic Vopěnka scheme holds for Σ_{n+2}-definable classes.*
  (3) *For every Σ_n-definable class A, there is a proper class of (weakly) virtually A-extendible cardinals.*
  (4) *There is a proper class of (weakly) virtually (Σ_n)-extendible cardinals.*
  (5) *There is a proper class of cardinals κ, such that for every Σ_n-correct cardinal λ above κ, there is a Σ_n-correct cardinal $\theta > \lambda$ and a virtual elementary embedding $j : V_\lambda \to V_\theta$ with critical point κ.*

Proof. We should like to emphasize that for none of the embeddings here do we insist that $\lambda < j(\kappa)$; we are using the weak forms only.

$(1 \to 2)$ Assume that the generic Vopěnka scheme holds for Π_{n+1}-definable classes and that $\mathcal{M}$ is a Σ_{n+2} definable class of structures, defined by $x \in \mathcal{M} \iff$

$\exists y\,\psi(x, y, p)$, where ψ is Π_{n+1} and p is a fixed parameter. For each structure $x \in \mathcal{M}$, let Y_x be the set of minimal-rank y witnessing that $x \in \mathcal{M}$ by the defining property, and form a new structure x^+ by adding Y_x to x as a new point, if necessary, and interpreting it as a new constant symbol. Let $\mathcal{N}$ be the class of structures x^+ obtained in this way. This class is Π_{n+1}-definable and any embedding of structures in $\mathcal{N}$ gives rise to an embedding of the corresponding structures in $\mathcal{M}$, as desired.

($2 \to 3$) Argue as in theorem 7. Assume that A is a Σ_n-definable class, and fix any γ. First, the collection of structures $\langle V_\lambda, \in, A \cap V_\lambda, \check{\alpha} \rangle_{\alpha \leq \gamma}$ that have no virtual embedding to some $\langle V_\theta, \in, A \cap V_\theta, \check{\alpha} \rangle_{\alpha \leq \gamma}$ is Π_{n+1}-definable, and so we get a virtual embedding from one of them to another, with critical point above γ. So for all sufficiently large λ, there are such embeddings with some critical point, and we may apply Fodor to find a single κ that works unboundedly often.

($3 \to 4$) This is immediate, since the Σ_n-truth predicate is Σ_n-definable.

($4 \to 5$) If κ is (weakly) virtually (Σ_n)-extendible and λ is Σ_n-correct, then we get $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \to \langle V_\theta, \in, A \cap V_\theta \rangle$ with critical point κ, where A is the Σ_n-truth predicate. Since V_λ can verify that $A \cap V_\lambda$ agrees with Σ_n-truth in V_λ, this will also be true for V_θ, and so θ must also be Σ_n-correct, as desired.

($5 \to 1$) Suppose that $\mathcal{M}$ is a Π_{n+1}-definable class of first-order structures in a common language, defined so that $x \in \mathcal{M} \iff \forall z\, \varphi(x, z, a)$, where φ has complexity Σ_n. Let κ be as in statement (5) and larger than the size of the language used for structures in $\mathcal{M}$. Let m be much larger than n and let $\lambda > \kappa$ be any Σ_m-correct ordinal. So it is also Σ_n-correct. By our assumption on κ, there is a virtual elementary embedding $j : V_\lambda \to V_\theta$ with critical point κ, where θ is Σ_n-correct. Let $M \in \mathcal{M}$ be any structure with rank amongst the κth rank to occur for structures in $\mathcal{M}$. This is observed correctly in V_λ. By the elementarity of the embedding, V_θ thinks that $j(M)$ is in $\mathcal{M}^{V_\theta}$, although V_θ may be wrong about this class. Since V_θ is Σ_n-correct, however, it is correct about φ, and from this it follows that $\mathcal{M} \cap V_\theta \subseteq \mathcal{M}^{V_\theta}$. In particular, V_θ knows $M \in \mathcal{M}^{V_\theta}$. Meanwhile, since $j(M)$ has the $j(\kappa)^{th}$ rank, it is not hard to see that $M \neq j(M)$, and since the critical point of j is above the language of M, it follows that $j \restriction M : M \to j(M)$ is an elementary embedding. By the absoluteness lemma 2, there will be such an embedding in any forcing extension collapsing M to be countable. So V_θ thinks that there is a virtual embedding between two elements of $\mathcal{M}^{V_\theta}$. By the elementarity of j, it follows that V_λ must also think this is true. But by the choice of λ, we know that V_λ is right about this. So we have verified this instance of the generic Vopěnka scheme. $\qquad\square$

The equivalencies proved in theorem 9 can be seen as uniformizing some of the results of [BGS17]. To avoid the bifurcation there into the two cases, such as getting from $\mathrm{gVP}(\Sigma_2)$ either a proper class of remarkable cardinals or a proper class of virtual rank-into-rank cardinals, what was needed was to drop the requirement that $\lambda < j(\kappa)$ for the embeddings, and then one gets a pure equivalence as above. The generic Vopěnka principle simply doesn't entitle one to embeddings j with the stronger property that $\lambda < j(\kappa)$.

4. A MODEL OF THE GENERIC VOPĚNKA PRINCIPLE IN WHICH THE ORDINALS ARE NOT MAHLO

We shall now construct a model of GBC plus the generic Vopěnka principle in which the ordinals are not Mahlo. The main idea will be to adapt Hamkins's

observation that one can easily separate the Vopěnka principle from the Vopěnka scheme.

Theorem 10 ([Ham]). *If the Vopěnka scheme holds, then there is a class-forcing forcing extension $V[C]$ where it continues to hold, yet, in which the Vopěnka principle fails and* ORD *is not Mahlo, although it remains definably Mahlo.*

The class forcing $\mathbb{P}$ is simply the standard forcing to kill ORD is Mahlo, the forcing to add a class club C avoiding the regular cardinals. Conditions in $\mathbb{P}$ are closed bounded sets containing no regular cardinals, ordered by end-extension. Over the GBC model in which it is defined, this forcing is $\leq\gamma$-distributive for every ordinal γ, because in fact the collection of conditions that reach above γ is a $\leq\gamma$-closed dense subclass of the forcing. Consequently, forcing with $\mathbb{P}$ over the model in which it is defined adds no new sets and preserves GBC. For these reasons, this forcing is amongst the nicest kind of class forcing that there is: over any GBC model, this forcing is definable; it has a definable forcing relation; it adds no new sets; and it preserves GBC. Since the generic class C itself witnesses that ORD is not Mahlo in the extension $V[C]$, it follows that the Vopěnka principle must fail there, but because the forcing adds no new sets, it preserves the Vopěnka scheme and consequently also the definable Mahloness of ORD.

We shall adapt the method here in order to prove that the generic Vopěnka scheme is also relatively consistent with the non-Mahloness of ORD.

Theorem 11. *Assume $0^{\#}$ exists in V. Then there is a class-forcing notion $\mathbb{P}$ definable in the constructible universe L, such that in any L-generic extension $L[C]$ by this forcing,* GBC *and the generic Vopěnka principle hold, yet* ORD *is not Mahlo.*

Proof. To begin, assume $0^{\#}$ exists, and let $\mathbb{P}$ be the class-forcing notion, as defined in L, to add a class club C avoiding the regular cardinals. In L, this forcing is $\leq\gamma$-distributive for every ordinal γ, and as we mentioned earlier, using this forcing over L adds no new sets; it has a definable forcing relation; and it preserves GBC. We view $L[C]$ as a GBC model having the classes that are definable from C, or in other words, definable in the structure $\langle L, \in, C\rangle$. Since C is a class club containing no regular cardinals, it follows that $L[C]$ thinks that ORD is not Mahlo.

Our use of class forcing is a bit unusual here, because although we assume $0^{\#}$ exists, we do not force over V, but rather only over L. We make the $0^{\#}$ assumption only in order to establish a certain density property for the forcing $\mathbb{P}$ in L, in order to know that it will succeed when used to force over L. Indeed, while the forcing $\mathbb{P}$ is very nice for forcing over L, meanwhile it is much less nice to force with $\mathbb{P}$ over V—this will definitely destroy GBC. The reason is that because of $0^{\#}$, the full model V already has a class club of L-regular cardinals, but these two class clubs cannot intersect. So if C is V-generic for $\mathbb{P}$, then in $V[C]$ we would be able to define a countable sequence cofinal in the ordinals, violating GBC. This is not a problem for our argument, however, because we shall make no reference to forcing over V and we shall never form the extension $V[C]$. Instead, our desired model is $L[C]$, which is a model of GBC, whose first-order part is L.

What remains is to prove that the generic Vopěnka principle holds in $L[C]$. For this, we make a density argument in the following lemma scheme. Since $\mathbb{P}$ is definable in L, for any ordinal θ, we may consider the analogue of the forcing $\mathbb{P}^{L_\theta}$ as defined inside L_θ. A set of ordinals c is L_θ-generic for $\mathbb{P}^{L_\theta}$, if it meets all dense

sets of $\mathbb{P}^{L_\theta}$ definable over L_θ, in the sense that every such dense set contains an initial segment of c.

Lemma 11.1. *Consider any ordinal δ and suppose n is a particular natural number of the meta-theory. Let $D_{\delta,n}$ be the collection of conditions $c \in \mathbb{P}$ for which there is an ordinal θ such that*

(1) $L_\theta \prec_{\Sigma_n} L$,
(2) $c \cap \theta$ *is L_θ-generic for $\mathbb{P}^{L_\theta}$, and*
(3) *in some forcing extension of L, there is an elementary embedding*

$$j : \langle L_\theta, \in, c \cap \theta \rangle \to \langle L_\theta, \in c \cap \theta \rangle$$

with critical point above δ.

Then $D_{\delta,n}$ is a definable dense subclass of $\mathbb{P}$ in L.

Proof. Fix any ordinal δ and any particular meta-theoretic natural number n (the lemma is a scheme as n changes). We want to show $D_{\delta,n}$ is dense in $\mathbb{P}$. Since the class $D_{\delta,n}$ gets smaller as n increases, we may imagine without loss that n is very large. Fix any condition $d \in \mathbb{P}$. We shall find $\bar{c} \in D_{\delta,n}$ extending d. Let κ_0 be any uncountable cardinal of V above δ and the supremum of d. Next, let

$$\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots < \kappa_\omega < \kappa_{\omega+1}$$

be the next $\omega + 2$ many successive Silver indiscernibles. Let θ be the least ordinal above κ_ω such that $L_\theta \prec L_{\kappa_{\omega+1}}$. Note that because $\kappa_{\omega+1}$ is a Silver indiscernible, this will imply $L_\theta \prec L$, and in particular, $L_\theta \prec_{\Sigma_n} L$. The ordinal θ has cofinality ω in L and is a limit cardinal there, and so we may find an ω-sequence of L-cardinals

$$\theta_0 < \theta_1 < \cdots < \theta_n < \cdots$$

cofinal in θ in L. We shall now construct $c \subseteq \theta$ end-extending d that is L_θ-generic for $\mathbb{P}^{L_\theta}$.

For each $k \in \omega$, let D_k be the intersection of all open dense subclasses of $\mathbb{P}^{L_\theta}$ that are Σ_k-definable in L_θ using parameters in L_{θ_k}. That is, we limit both the complexity of the definition and the space of parameters. Since Σ_k-truth is definable, the model L_θ has uniform definable access to its Σ_k-definable classes. And since there are only θ_k many parameters involved, the model L_θ may therefore enumerate its Σ_k-definable classes with parameters in L_{θ_k} in a single θ_k-sequence. Since the forcing $\mathbb{P}^{L_\theta}$ is $\leq\theta_k$-distributive, it follows that D_k is an open dense subclass of $\mathbb{P}^{L_\theta}$.

The classes D_k provide a countable collection of dense subclasses that suffice for L_θ-genericity, since any subclass of $\mathbb{P}^{L_\theta}$ that is definable in L_θ will be contained in some D_k. By the usual diagonalization procedure, therefore, we may build a set $c \subseteq \theta$ extending d by successively extending it so as to meet each D_k in turn. It follows that $c \subseteq \theta$ is L_θ-generic for $\mathbb{P}^{L_\theta}$. Since this diagonalization construction can be carried out inside L, we may find such L_θ-generic sets c in L. Henceforth, let c be the L-least such L_θ-generic set for $\mathbb{P}^{L_\theta}$ extending d. Note that c is definable in L from parameters d, κ_ω, and $\kappa_{\omega+1}$.

Let $j : L \to L$ be the elementary embedding generated by a shift of the indiscernibles κ_n, so that $j(\kappa_n) = \kappa_{n+1}$ for $n \in \omega$, and fixing all other indiscernibles. Since we chose κ_0 to be an uncountable cardinal of V, it follows that it is a limit of smaller indiscernibles, which generate L_{κ_0}. It follows that the critical point of j is precisely κ_0 and in particular, $j(d) = d$. Since κ_ω and $\kappa_{\omega+1}$ are fixed by j, it follows

similarly that $j(\theta) = \theta$. Since c was defined by those fixed points, it also follows that $j(c) = c$. Thus, the restriction of j to L_θ gives an elementary embedding

$$j : \langle L_\theta, \in, c \rangle \to \langle L_\theta, \in, c \rangle.$$

By the absoluteness lemma 2, it follows that in some forcing extension of L, there is an elementary embedding

$$j^* : \langle L_\theta, \in, c \rangle \to \langle L_\theta, \in, c \rangle$$

with critical point κ_0. Finally, let $\bar{c} = c \cup \{\theta\}$ be the closure of this set, and observe that this is a condition in $\mathbb{P}$ precisely because θ is singular in L. We have therefore verified all the necessary requirements to conclude that $\bar{c} \in D_{\delta,n}$, and since $\bar{c}$ extends d, we have therefore proved that this class is dense, establishing lemma 11.1. □

Using the lemma, we shall now complete the proof of theorem 11. Fix any proper class $\mathcal{M}$ in $L[C]$ of first-order structures in a common language $\mathcal{L}$. Since we have included only the C-definable classes in $L[C]$, we may assume that $\mathcal{M}$ is defined by some Σ_m-formula $\psi(x, a, C)$ with class parameter C and set parameter $a \in L$. Let δ be above the rank of the language $\mathcal{L}$ and the parameter a, and let n be much larger than m. By the lemma, there is some ordinal θ such that the corresponding initial segment of C is in $D_{\delta,n}$. So there is, in some forcing extension of L, an elementary embedding

$$j : \langle L_\theta, \in, C \cap \theta \rangle \to \langle L_\theta, \in, C \cap \theta \rangle$$

with critical point κ above δ.

We claim next that $\langle L_\theta, \in, C \cap \theta \rangle \prec_{\Sigma_m} \langle L, \in, C \rangle$. To see this, suppose that $\langle L_\theta, \in, C \cap \theta \rangle \models \varphi(b)$ for some Σ_m-formula $\varphi(x)$. This must be forced over L_θ by some condition c, an initial segment of $C \cap \theta$. By the choice of n as much larger than m, what we had meant was that it should be large enough to express this forcing relation, and so since $L_\theta \prec_{\Sigma_n} L$ it follows that c forces $\varphi(b)$ also over L. So $\langle L, \in, C \rangle \models \varphi(b)$, as desired.

It follows that the definition of the class $\mathcal{M}$ is absolute to $\langle L_\theta, \in, C \cap \theta \rangle$. Let M be the κth element of $\mathcal{M}$ in the L-order. It follows that $j(M)$ is the $j(\kappa)^{th}$ element, which is identified correctly by L_θ, and consequently $M \neq j(M)$. The restriction $j \upharpoonright M : M \to j(M)$ is an elementary embedding between distinct elements of $\mathcal{M}$, and so we have verified this instance of the generic Vopěnka principle in $L[C]$, as desired, completing the proof of theorem 11. □

It was convenient to use $0^\sharp$ in the previous argument, in order to know that $D_{\delta,n}$ is dense in $\mathbb{P}$, but it is natural to inquire whether this use can be weakened or eliminated. We shall discuss this in section 6.

5. A MODEL OF THE GENERIC VOPĚNKA SCHEME IN WHICH THE ORDINALS ARE NOT Δ_2-MAHLO

We should now like to sharpen the result of the previous section by performing further forcing so as to make the class club C definable, in fact Δ_2-definable, while preserving the generic Vopěnka scheme. The result will be a model of GBC plus the generic Vopěnka scheme in which there are no Σ_2-reflecting cardinals (inaccessible Σ_2-correct cardinals). Consequently, there will also be no remarkable cardinals.

Theorem 12. *Assume $0^{\#}$ exists in V. Then there is a definable class-forcing notion in L, such that in the corresponding L-generic extension,* GBC *holds, the generic Vopěnka scheme holds, but* ORD *is not definably Mahlo. Indeed, in this model there is a Δ_2-definable class club avoiding the regular cardinals.*

Proof. The forcing will be a two-step iteration, although this forcing can also be viewed as a single-step class forcing. First, we force with $\mathbb{P}$ as in section 4 to add a class club C avoiding the regular cardinals of L. We proved in theorem 11, under the assumption that $0^{\sharp}$ exists, an assumption we also have here, that the generic Vopěnka principle holds in $L[C]$, but the ordinals are not Mahlo there. Next, we force over $L[C]$ with the forcing $\mathbb{Q}$ that codes the class C into the continuum pattern. Specifically, $\mathbb{Q}$ is the ORD-length Easton product forcing, as defined in $L[C]$, which forces violations of the GCH exactly at the successor cardinals of the elements of C. This forcing is very mild in terms of class forcing. It is definable; it factors in the Easton manner at every element of C into the product of small forcing and highly closed forcing; it is therefore progressively closed; and consequently, the forcing $\mathbb{Q}$ preserves GBC and has a definable forcing relation satisfying the forcing theorems.

The two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ can be viewed as a one-step non-iterative class forcing, simply by viewing the conditions as pairs (d, q), where d is a closed bounded set in L avoiding the L-regular cardinals, and q is a condition in the corresponding Easton-support product forcing to code d into the GCH pattern. This is dense in the iteration forcing, consisting of pairs $(d, \dot{q})$ where $d \in \mathbb{P}$ and $\dot{q}$ is a $\mathbb{P}$-name for a condition in $\dot{\mathbb{Q}}$, because the $\mathbb{P}$ forcing adds no new sets and so one may simply strengthen the first coordinate so as to decide the value of $\dot{q}$ in the second coordinate. Further, if $\dot{q}$ is any $\mathbb{P}$-name for a condition in $\dot{\mathbb{Q}}$, then there is a set-sized maximal antichain of conditions $d \in \mathbb{P}$ that decide the value of $\dot{q}$ as $\check{q}$ for some $q \in L$. Thus, any name for the iteration forcing can be transformed to a name for the non-iterative combined forcing we mentioned, and therefore the two versions of the forcing give rise to the same forcing extensions. So we needn't think of it as an iteration at all, and the combined forcing inherits the nice properties of $\mathbb{P}$ and $\mathbb{Q}$ and will preserve GBC and have a definable forcing relation satisfying the forcing theorem and so on.

Having added the generic class club C, let $G \subseteq \mathbb{Q}$ be $L[C]$-generic, and we consider the forcing extension $L[C][G]$. This is a GBC model, whose sets are those added by the $\mathbb{Q}$ forcing and whose classes are those definable from the generic classes C and G. But actually, since ultimately we aim merely to construct a ZFC model, let us use the notation $L[G]$ to refer only to the first-order part of this model, having only the sets and not the classes. That is, we take

$$L[G] = \{\, \tau_G \mid \tau \text{ is a } \mathbb{Q}\text{-name in } L \,\}$$

to consist of the interpretations via G of the $\mathbb{Q}$-names in L. This is a model of ZFC, since it the first-order part of a model of GBC.

The coding forcing $\mathbb{Q}$ ensures that the successor cardinals of the elements of C can be identified in a Δ_2 manner in the extension $L[G]$, and so the class C is Δ_2-definable in $L[G]$. Thus, in $L[G]$ we have a parameter-free Δ_2-definable class club avoiding the regular cardinals. So ORD is definitely not Mahlo there, and not even Δ_2-Mahlo.

It remains to argue that $L[G]$ satisfies the generic Vopěnka scheme. Let $\mathbb{Q}_\theta$ be the factor of $\mathbb{Q}$ consisting of the forcing only on the coordinates below θ, that is, with

conditions having support contained in θ. Note that if $C \cap \theta$ is unbounded in θ, but θ is singular, then because of the nature of the Easton support, the forcing $\mathbb{Q}$ will have conditions with support unbounded in θ. In particular, $\mathbb{Q}_\theta$ also has such conditions, and so we cannot view $\mathbb{Q}_\theta$ as class forcing over the structure $\langle L_\theta, \in, C \cap \theta \rangle$. That structure simply doesn't include all the conditions of $\mathbb{Q}_\theta$. Nevertheless, since $C \cap \theta$ is a set in L, it follows that $\mathbb{Q}_\theta$ is a forcing notion in L. If $G \subseteq \mathbb{Q}$ is L-generic, let us denote by G_θ the restriction of G to $\mathbb{Q}_\theta$.

Lemma 12.1. *Suppose that δ is an ordinal and n is a particular natural number of the meta-theory. Then there is an ordinal θ for which*

(1) $L_\theta \prec_{\Sigma_n} L$,
(2) $C \cap \theta$ *is generic for dense subsets of $\mathbb{P}^{L_\theta}$ that are Σ_n-definable in L_θ,*
(3) $G_\theta \cap \mathbb{Q}^{L_\theta[C \cap \theta]}$ *is generic for dense subsets of $\mathbb{Q}^{L_\theta[C \cap \theta]}$ that are Σ_n-definable in $L_\theta[C \cap \theta]$, and*
(4) *in a forcing extension of $L[G]$, there is an elementary embedding $j : L_\theta[G_\theta] \to L_\theta[G_\theta]$ with critical point above δ.*

Proof. This is a lemma scheme, taken as n varies. But since the statement becomes harder as n increases, we may assume without loss that n is very large. Fix n and any ordinal δ. Let us explain a little further about what we mean in statements (3) and (4). The forcing notion $\mathbb{Q}_\theta$ is the factor of $\mathbb{Q}$ at coordinates up to θ, but since θ is singular, this allows for conditions with unbounded support. The forcing notion $\mathbb{Q}^{L_\theta[C \cap \theta]}$, in contrast, is defined just as $\mathbb{Q}$, except internally to $L_\theta[C \cap \theta]$, which means that we now take only the bounded-support conditions. Since G is fully $L[C]$-generic for $\mathbb{Q}$, it follows easily that $G_\theta \subseteq \mathbb{Q}_\theta$ is fully $L[C \cap \theta]$-generic for $\mathbb{Q}_\theta$. What we are claiming in statement (3), however, is something a bit more, namely, that for this particular θ, the bounded-support fragment of G_θ, meaning $G_\theta \cap \mathbb{Q}^{L_\theta[C \cap \theta]}$, is generic for dense subsets of $\mathbb{Q}^{L_\theta[C \cap \theta]}$ that are Σ_n-definable in $L_\theta[C \cap \theta]$. In statement (4), the structure $L_\theta[G_\theta]$ can therefore be viewed either as the extension arising from the bounded-support forcing, or as $V_\theta^{L[G]}$, since it is not difficult to see that these are the same (for $n \geq 1$), since every new set is added by a stage.

To prove the lemma, since the statement of the lemma is expressible in the forcing language using the canonical names for the generic filters, it suffices to show that no condition forces the negation of the statement.[2] Fix any condition $(d, r) \in \mathbb{P} * \mathbb{Q}$ in our class forcing.

We proceed at first as in lemma 11.1. Let κ_0 be an uncountable cardinal of V above the ranks of d and r and also above δ. Let

$$\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots < \kappa_\omega < \kappa_{\omega+1}$$

be the next $\omega + 2$ many successive Silver indiscernibles.

Let $d^+ = d \cup \{\kappa_\omega + 1\}$, which jumps over all the κ_n's and is a condition in $\mathbb{P}$. To find θ, suppose temporarily that d^+ agrees with C and $r \in G$ (if this is not the case, replace them with generic filters that do have this property). In the extension $L[C][G]$, we have ZFC in the language with predicates for C and G.

---

[2]Note that if we had stated in the lemma that $L_\theta \prec L$, which is what we shall achieve in the proof, using the $0^\sharp$ assumption, then the statement of the lemma would not be expressible in L in the forcing language, since L cannot express the property $L_\theta \prec L$. It shall be enough for our purpose, however, to require only $L_\theta \prec_{\Sigma_n} L$ and state the lemma as a scheme over natural numbers n in the meta-theory.

And C is certainly generic for Σ_n-definable subclasses of $\mathbb{P}$ and G is generic for dense subclasses of $\mathbb{Q}$ that are Σ_n-definable in $L[C]$. By the reflection theorem, therefore, there must be an ordinal θ of cofinality ω reflecting these facts to θ, so that $L_\theta \prec_{\Sigma_n} L$ and $C \cap \theta$ is Σ_n-generic and $G \cap \mathbb{Q}^{L[C\cap\theta]}$ is Σ_n-generic for $\mathbb{Q}^{L_\theta[C\cap\theta]}$ over $L_\theta[C \cap \theta]$. So there must be a condition (c, q) extending (d^+, r) and forcing that $C \cap \theta$ and $\dot{G}_\theta$ are like this. Note that no part of c or q above θ can matter for this property, and so we may assume $\sup(c) = \theta$ and $q \in \mathbb{Q}_\theta$ as defined from c.

Now, throw away the previous actual C and G, and let θ be the least ordinal with cofinality ω for which $\kappa_\omega < \theta < \kappa_{\omega+1}$ and there is a condition (c, q) in the forcing $\mathbb{P} * \dot{\mathbb{Q}}$ forcing the properties we have mentioned. And let (c, q) be the L-least such condition extending (d^+, r). So these conditions are definable from the parameters (d, r), κ_ω, and $\kappa_{\omega+1}$.

In V, let $j : L \to L$ be the elementary embedding generated by shifting the indiscernibles κ_n, so that $j(\kappa_n) = \kappa_{n+1}$ for all $n \in \omega$, but all other indiscernibles are fixed. Since κ_0 is an uncountable cardinal of V, it follows as before that the critical point of j is exactly κ_0, and so j fixes both d and r. Since it also fixes κ_ω and $\kappa_{\omega+1}$, it follows by the definability considerations that it fixes θ, c and q.

Consider now just the forcing $\mathbb{Q}_\theta$ defined by c in L. We claim that in a suitable forcing extension of V, we will be able to find an L-generic filter $H_\theta \subseteq \mathbb{Q}_\theta$ with $q \in H_\theta$, such that j lifts to

$$j : L[H_\theta] \to L[H_\theta]$$

and such that $j(H_\theta) = H_\theta$. To begin with this, note that since c is empty on the interval $[\kappa_0, \kappa_\omega)$, it follows that $\mathbb{Q}_\theta$ is trivial on this interval. So the coding forcing $\mathbb{Q}_\theta$ factors as very small forcing $\mathbb{Q}_\theta^{\text{small}}$ to code c below κ_0, followed by the coding forcing $\mathbb{Q}_\theta^{\text{tail}}$, which is above κ_ω. So we have

$$\mathbb{Q}_\theta \cong \mathbb{Q}_\theta^{\text{tail}} \times \mathbb{Q}_\theta^{\text{small}},$$

where $\mathbb{Q}_\theta^{\text{tail}}$ is $\leq\kappa_\omega$-closed in L and $\mathbb{Q}_\theta^{\text{small}}$ has size less than κ_0 in L. We may similarly factor the condition q as $q = q^{\text{small}} \times q^{\text{tail}}$.

We claim that in order for a filter $H \subseteq \mathbb{Q}_\theta^{\text{tail}}$ to be L-generic, it suffices for H to meet all the open dense subsets of this forcing that are definable in L using only indiscernible parameters not smaller than κ_ω. To see this, suppose $D \subseteq \mathbb{Q}_\theta^{\text{tail}}$ is an arbitrary open dense subset of the forcing in L. Since every set in L is definable from the Silver indiscernibles, there is a formula φ and indiscernible parameters $\vec{\kappa}_{\text{small}}, \vec{\kappa}_{\text{tail}}$, such that

$$p \in D \;\leftrightarrow\; \varphi(p, \vec{\kappa}_{\text{small}}, \vec{\kappa}_{\text{tail}}),$$

where $\vec{\kappa}_{\text{small}}$ are below κ_ω and $\vec{\kappa}_{\text{tail}}$ are not. Let

$$D_{\vec{\alpha}} = \{p \in \mathbb{Q}_\theta^{\text{tail}} \mid \varphi(p, \vec{\alpha}, \vec{\kappa}_{\text{tail}})\},$$

where we allow arbitrary parameters $\vec{\alpha}$ in place of $\vec{\kappa}_{\text{small}}$. Let $\bar{D}$ be the intersection of all $D_{\vec{\alpha}}$ that happen to be open and dense in $\mathbb{Q}_\theta^{\text{tail}}$, ranging over all $\vec{\alpha} < \kappa_\omega$. Since the forcing $\mathbb{Q}_\theta^{\text{tail}}$ is $\leq\kappa_\omega$-closed, it follows that $\bar{D}$ is dense open. But furthermore, by its nature, $\bar{D}$ is definable from $\vec{\kappa}_{\text{tail}}$, which are Silver indiscernibles not less than κ_ω. If a filter meets $\bar{D}$, then it also meets D, since $\bar{D} \subseteq D$.

Notice furthermore that if a condition $p \in \mathbb{Q}_\theta^{\text{tail}}$ is definable from indiscernible parameters other than the κ_n's and a dense set D is definable from non-κ_n indiscernible parameters, then the L-least element in D extending p will also be definable

from such parameters. Moreover, any such condition is a fixed point $j(p) = p$, since j fixes all indiscernibles except the κ_n.

Let's go now to a forcing extension $V[K]$ by collapsing θ^+ to become countable. We shall argue that in $V[K]$ we can construct the desired filter H_θ^{tail}. The extension $V[K]$ sees that there are only countably many subsets of $\mathbb{Q}_\theta^{\text{tail}}$ in L, and it can observe which of them are open and dense and definable in L using only indiscernibles not less than κ_ω. If $\langle D_n \mid n < \omega \rangle$ is the enumeration of these, then let q_0 be the L-least condition in D_0 extending q^{tail}. Similarly, let q_{n+1} be the L-least extension of q_n in D_{n+1}. Thus, if H_θ^{tail} is the filter generated by the conditions q_n, it will be L-generic for $\mathbb{Q}_\theta^{\text{tail}}$ and contain the condition q^{tail}. Further, since each q_n is definable from indiscernibles below κ_ω and above, it follows by the observation of the previous paragraph that $j(q_n) = q_n$ for each n. Thus, $j \text{ " } H_\theta^{\text{tail}} = H_\theta^{\text{tail}}$ and so we fulfill the lifting criterion for the embedding $j : L \to L$, which now lifts to $j : L[H_\theta^{\text{tail}}] \to L[H_\theta^{\text{tail}}]$, with $j(H_\theta^{\text{tail}}) = H_\theta^{\text{tail}}$.

And if H_θ^{small} is any further $L[H_\theta^{\text{tail}}]$-generic filter for $\mathbb{Q}_\theta^{\text{small}}$, then since this forcing is below the critical point of j it also lifts easily to $j : L[H_\theta] \to L[H_\theta]$, where $H_\theta = H_\theta^{\text{small}} \times H_\theta^{\text{tail}} \subseteq \mathbb{Q}_\theta$, with $j(H_\theta) = H_\theta$. We may assume $q^{\text{small}} \in H_\theta^{\text{small}}$ and therefore $q \in H_\theta$. So in a forcing extension of V, we have found the kind of embedding we claimed.

It follows by the absoluteness lemma 2 that in a forcing extension of L, there is an L-generic filter $H_\theta \subseteq \mathbb{Q}_\theta$ containing the condition q and an elementary embedding $j^* : L_\theta[H_\theta] \to L_\theta[H_\theta]$ with critical point above δ. We also have $L_\theta \prec L$. Now we can extend c and H_θ fully to L-generic filters C and H for the forcing $\mathbb{P} * \dot{\mathbb{Q}}$. So finally, because $(c, q) \in C * H$, we have that $C \cap \theta$ is Σ_n-generic for $\mathbb{P}^{L_\theta}$ over L_θ and $H_\theta \cap \mathbb{Q}^{L_\theta[C \cap \theta]}$ is Σ_n-generic over $L_\theta[C \cap \theta]$.

Since we have attained precisely the properties stated in the lemma, but for a generic filter containing the original condition (d, r), our argument shows that (d, r) could not have forced that this situation does not happen. And so lemma 12.1 is proved.   □

We now continue with the proof of theorem 12. In order to prove that the generic Vopĕnka scheme holds in $L[G]$, suppose that $\mathcal{M} = \{ x \mid \psi(x, a) \}$ is a definable class of first-order structures in a common language in $L[G]$, defined by the Σ_m-formula ψ with parameter a. Let n be much larger than m, and let δ be large enough so that the common language of the structures in $\mathcal{M}$ and the parameter a have rank less than δ. By lemma 12.1, there is an ordinal θ such that $L_\theta \prec_{\Sigma_n} L$, with $C \cap \theta$ being Σ_n-generic for $\mathbb{P}^{L_\theta}$ over L_θ and $G \cap \mathbb{Q}^{L_\theta[C \cap \theta]}$ being Σ_n-generic for $\mathbb{Q}^{L_\theta[C \cap \theta]}$ over $L_\theta[C \cap \theta]$, and such that, in some forcing extension of L, there is an embedding $j : L_\theta[G_\theta] \to L_\theta[G_\theta]$ with critical point above δ.

Since n is much larger than m, it follows by the definability of the forcing relation that $\mathcal{M}$ is absolute to $L_\theta[G_\theta]$. Let M be a structure in $\mathcal{M}$ of the κth rank occurring in this class. So $j(M)$ is in $\mathcal{M}$ of the $j(\kappa)^{th}$ rank, and so $M \neq j(M)$. Since the common language of the structures in $\mathcal{M}$ is fixed by j, it follows that $j \upharpoonright M : M \to j(M)$ is an elementary embedding. So we have witnessed the existence of a virtual elementary embedding between distinct structures in $\mathcal{M}$, and thereby verified this instance of the generic Vopĕnka scheme in $L[G]$, as desired. This completes the proof of theorem 12.   □

Corollary 13. *If $0^\sharp$ exists, then there is a class-forcing extension $L[G]$ of the constructible universe in which the generic Vopěnka principle holds, but there are no Σ_2-reflecting cardinals and hence no remarkable cardinals.*

Proof. Consider the model $L[G]$ constructed in theorem 12. In that model, we have the generic Vopěnka principle, yet there is a parameter-free definable class club C containing no regular cardinals. Since the definition of C uses no parameters, it follows that if κ were Σ_2-reflecting, then $C \cap \kappa$ would be unbounded in κ and consequently $\kappa \in C$, contrary to the assumption that there are no regular cardinals in C. So there can be no Σ_2-reflecting cardinals. And since every remarkable cardinal is Σ_2-reflecting, it follows similarly that there can be no remarkable cardinals in $L[G]$.   □

This provides the negative answer to question 1.

Corollary 14. *It is relatively consistent with* GBC *that every class A admits a (weakly) virtually A-extendible cardinal (and so the generic Vopěnka principle holds), but no class A admits a (strongly) virtually A-extendible cardinal.*

Proof. If κ is (strongly) virtually extendible, then κ is clearly Σ_2-reflecting, since the targets $j(\kappa)$ can be chosen as high in V as desired, thereby capturing any witness of a Σ_2 assertion. In this case, κ would be Σ_2-reflecting, contrary to the existence of a parameter-free Δ_2-definable club containing no regular cardinals in the model of theorem 12.   □

6. CAN WE WEAKEN OR ELIMINATE THE $0^\sharp$ ASSUMPTION?

In our main theorem, we had assumed the existence of $0^\sharp$ in V in order to attain the generic Vopěnka principle and scheme in a class-forcing extension $L[G]$ of the constructible universe, in which the ordinals were not Δ_2-Mahlo. In our model, there was a parameter-free Δ_2-definable class club C containing no regular cardinals. It follows that there can be no Σ_2-reflecting cardinals and therefore also no remarkable cardinals, since every remarkable cardinal is Σ_2-reflecting.

It is natural to inquire whether our use of $0^\sharp$ can be weakened or eliminated. Perhaps it is natural for one to hope to prove that if the generic Vopěnka principle holds in L, then it continues to hold in our forcing extension $L[G]$. Doing so would not only improve the theorem, by weakening the hypotheses, but it would also address the possibly unnecessary meta-mathematical aspect of the argument, whereby we assume $0^\sharp$ in V, but then force over L.

But alas, if these hypotheses are consistent, then that will not be possible. The reason is that the generic Vopěnka principle, if consistent, has a strictly weaker consistency strength than the theory we obtain in $L[G]$, namely, the generic Vopěnka principle plus ORD is not Δ_2-Mahlo, which implies that there are no Σ_2-reflecting cardinals and therefore no remarkable cardinals. By the bifurcation result of Bagaria, Gitman and Schindler [BGS17], since there are no remarkable cardinals in our model, then there must be a proper class of virtually rank-into-rank cardinals, and the consistency strength of this is strictly higher than the generic Vopěnka principle itself (but less than $0^\sharp$).

We find it quite reasonable to expect to prove our theorem starting from a model with a suitable proper class of virtually rank-into-rank cardinals, replacing our indiscernibility embeddings with those virtual rank-into-rank embeddings. We leave the details of this to another project.

Meanwhile, it is easy to see that the existence of $0^\sharp$ has a strictly higher consistency strength than is necessary for our conclusion, where we have the generic Vopěnka principle with a Δ_2-definable class club. The reason is that our conclusion is expressible in the first-order language of set theory and therefore reflects to the initial segments of L, cut off at any Silver indiscernible ordinal. So under $0^\sharp$, we are able to construct transitive models of our target theory, and so $0^\sharp$ is strictly stronger than necessary.

7. Tying up a loose end

We'd like to conclude our paper by tying up a certain loose end, by proving that some various large cardinals properties that have been considered in the literature in connection with the Vopěnka scheme are actually equivalent.

Recall that a cardinal κ is Σ_n-*correct*, if $V_\kappa \prec_{\Sigma_n} V$. Let $C^{(n)}$ be the class of Σ_n-correct cardinals. Bagaria [Bag12] proved that $\text{VP}(\Sigma_{n+2})$ holds precisely when there is a proper class of $C^{(n)}$-extendible cardinals, where a cardinal κ is $C^{(n)}$-extendible (in Bagaria's sense), if for every $\lambda > \kappa$, there is an elementary embedding $j : V_\lambda \to V_\theta$ with critical point κ and $\lambda < j(\kappa) \in C^{(n)}$. It is easy to see that every extendible cardinal κ is $C^{(1)}$-extendible, because the $j(\kappa)$ of an extendibility embedding j is inaccessible.

An observant reader will have noticed, however, that there is a possible collision in the terminology, since we have two possibly different conceptions of what it means to be $C^{(n)}$-extendible. Namely, on the one hand, we have the notion of $C^{(n)}$-extendible in the sense of Bagaria, which we just defined in the previous paragraph. On the other hand, we have the notion of $C^{(n)}$-extendible in the sense of Hamkins, defined in the introduction of this article, that is, A-extendible when A happens to be the class $C^{(n)}$. There is also another concept of relative extendibility, due to Bagaria, namely, κ is $C^{(n)+}$-*extendible*, if for every $\lambda \in C^{(n)}$ above κ there is $\theta \in C^{(n)}$ and an elementary embedding $j : V_\lambda \to V_\theta$ with critical point κ and $\lambda < j(\kappa)$. Bagaria proved that the least $C^{(n)}$-extendible cardinal is also $C^{(n)+}$-extendible [Bag12]. And recall from section 3 our definition that κ is (Σ_n)-*extendible*, if it is A-extendible when A is the Σ_n-truth predicate.

We shall now happily prove that all these notions coincide, and so there is actually no collision in the terminology after all!

Theorem 15. *The following are equivalent for any cardinal κ and any particular finite $n \geq 1$.*

(1) κ is $C^{(n)}$-*extendible in the sense of Bagaria, so that for every $\lambda > \kappa$, there is an elementary embedding $j : V_\lambda \to V_\theta$ with critical point κ and $\lambda < j(\kappa) \in C^{(n)}$.*

(2) κ is $C^{(n)}$-*extendible in the sense of Hamkins, that is, A-extendible where A is the class $C^{(n)}$, so that for every λ there is an elementary embedding $j : \langle V_\lambda, \in, C^{(n)} \cap V_\lambda \rangle \to \langle V_\theta, \in, C^{(n)} \cap V_\theta \rangle$ with critical point κ and $\lambda < j(\kappa)$.*

(3) κ is A-*extendible in the sense of Hamkins, for every Σ_n-definable class A, allowing parameters in V_κ.*

(4) κ is (Σ_n)-*extendible in the sense of Hamkins, meaning that it is A-extendible where A is a Σ_n-truth predicate.*

(5) κ is $C^{(n)+}$-extendible in the sense of Bagaria, so that for every $\lambda \in C^{(n)}$ above κ, there is an elementary embedding $j : V_\lambda \to V_\theta$ for some $\theta \in C^{(n)}$ with critical point κ and $\lambda < j(\kappa)$.

Proof. This is a theorem scheme, a separate theorem for each finite natural number n in the meta-theory.

$(5 \to 4)$ If $j : V_\lambda \to V_\theta$ for $\lambda, \theta \in C^{(n)}$, then both V_λ and V_θ are correct about Σ_n-truth, and so j is elementary in the language with a predicate for Σ_n-truth. In other words, κ is (Σ_n)-extendible in the sense of Hamkins.

$(4 \to 3)$ If $j : V_\lambda \to V_\theta$ is elementary with respect to the predicate for Σ_n-truth, then it is also elementary with respect to any Σ_n-definable class, allowing parameters in V_κ, since that class A is definable from the Σ_n-truth predicate.

$(3 \to 2)$ Immediate, since $C^{(n)}$ is Π_n-definable, and being elementary for the complement of a predicate is the same as being elementary for the predicate itself.

$(2 \to 1)$ If κ is $C^{(n)}$-extendible in the sense of Hamkins, then in particular, κ must be a limit point of $C^{(n)}$ and hence also an element of $C^{(n)}$. So if $j : \langle V_\lambda, \in, C^{(n)} \cap V_\lambda \rangle \to \langle V_\theta, \in, C^{(n)} \cap V_\theta \rangle$ has critical point κ, it follows that $j(\kappa) \in C^{(n)}$, as needed for Bagaria's notion.

$(1 \to 5)$ Assume that κ is $C^{(n)}$-extendible in the sense of Bagaria, and consider any $\lambda \in C^{(n)}$ above κ. Since any $\Sigma_2(C^{(n)})$-assertion reflects below κ, it follows that $\kappa \in C^{(n+2)}$. Let $\bar{\lambda} \in C^{(n+2)}$ be larger than λ. By the extendibility assumption, we get an elementary embedding $j : V_{\bar{\lambda}} \to V_{\bar{\theta}}$ for some $\bar{\theta}$ with critical point κ and $\bar{\lambda} < j(\kappa) \in C^{(n)}$. It follows that $V_{\bar{\theta}}$ is correct about $C^{(n)}$ below $j(\kappa)$, although it may be possibly wrong about $C^{(n)}$ above $j(\kappa)$. Let $\theta = j(\lambda)$, so that $j \upharpoonright V_\lambda : V_\lambda \to V_\theta$ is an elementary embedding, and furthermore $j \upharpoonright V_\lambda \in V_{\bar{\theta}}$. Since $\lambda \in C^{(n)}$ and $V_{\bar{\lambda}}$ knows this, it follows that $\theta = j(\lambda)$ is in $(C^{(n)})^{V_{\bar{\theta}}}$, even though $V_{\bar{\theta}}$ may disagree with V about $C^{(n)}$. So $V_{\bar{\theta}}$ thinks that "there is $\theta \in C^{(n)}$ and an elementary embedding h from V_λ to V_θ, with critical point κ and $\lambda < h(\kappa)$." Since the class $C^{(n)}$ is Π_n-definable, this is a Σ_{n+1}-expressible statement about κ and λ, which is true in $V_{\bar{\theta}}$. Since $V_{\bar{\lambda}}$ knows that $\kappa \in C^{(n+2)}$, it follows that $V_{\bar{\theta}}$ thinks that $j(\kappa)$ is in $C^{(n+2)}$—although it could be wrong about this—and so the Σ_{n+1} statement about κ and λ reflects from $V_{\bar{\theta}}$ to $V_{j(\kappa)}$. So $V_{j(\kappa)}$ thinks there is an ordinal $\theta' \in C^{(n)}$ and elementary embedding $h : V_\lambda \to V_{\theta'}$ with critical point κ and $\lambda < h(\kappa)$. Since $V_{j(\kappa)}$ is right about $C^{(n)}$, this verifies that κ is $C^{(n)+}$-extendible in V, as desired. $\square$

The argument for $(1 \to 5)$ can be seen as the non-generic analogue of the argument $(6 \to 0)$ in theorem 9.

References

[Bag12]   Joan Bagaria. $C^{(n)}$-cardinals. *Arch. Math. Logic*, 51(3-4):213–240, 2012.

[BGS17]   Joan Bagaria, Victoria Gitman, and Ralf Schindler. Generic Vopěnka's Principle, remarkable cardinals, and the weak Proper Forcing Axiom. *Arch. Math. Logic*, 56(1-2):1–20, 2017.

[BHTU]   Joan Bagaria, Joel David Hamkins, Konstantinos Tsaprounis, and Toshimichi Usuba. Superstrong and other large cardinals are never Laver indestructible. *to appear in Archive for Mathematical Logic (special issue in honor of Richard Laver)*.

[GS]   Victoria Gitman and Ralf Schindler. Virtual large cardinals. Manuscript.

[Ham]   Joel David Hamkins. The Vopěnka principle is inequivalent to but conservative over the Vopěnka scheme. manuscript under review.

(V. Gitman) THE CITY UNIVERSITY OF NEW YORK, CUNY GRADUATE CENTER, MATHEMATICS PROGRAM, 365 FIFTH AVENUE, NEW YORK, NY 10016, USA
  E-mail address: `vgitman@nylogic.org`
  URL: `http://boolesrings.org/victoriagitman`

(J. D. Hamkins) MATHEMATICS, PHILOSOPHY, COMPUTER SCIENCE, THE GRADUATE CENTER OF THE CITY UNIVERSITY OF NEW YORK, 365 FIFTH AVENUE, NEW YORK, NY 10016 & MATHEMATICS, COLLEGE OF STATEN ISLAND OF CUNY
  E-mail address: `jhamkins@gc.cuny.edu`
  URL: `http://jdh.hamkins.org`